# Relations and Functions 1

## Top Concepts in Relations

**1. Introduction to Relation and no. of relations**

- A relation R between two non-empty sets A and B is a subset of their Cartesian product A × B.
- If A = B, then the relation R on A is a subset of A × A.
- The total number of relations from a set consisting of m elements to a set consisting of n elements is $2^{mn}$.
- If (a, b) belongs to R, then a is related to b and is written as 'a R b'. If (a, b) does not belong to R, then a is not related to b and it is written as 'a $\not R$ b'.

**2. Co-domain and Range of a Relation**

Let R be a relation from A to B. Then the 'domain of' $R' \subset A$ and the 'range of $R' \subset B$. Co-domain is either set B or any of its superset or subset containing range of R.

**3. Types of Relations**

A relation R in a set A is called an empty relation if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

A relation R in a set A is called a universal relation if each element of A is related to every element of A, i.e., R = A × A.

**4. A relation R on a set A is called:**

**a.** Reflexive, if (a, a) ∈ R for every a ∈ A.

**b.** Symmetric, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$ for all $a_1, a_2 \in A$.

**c.** Transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that (a1, a3) ∈ R for all $a_1, a_2, a_3 \in A$.

**5. Equivalence Relation**

- A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.
- An empty relation R on a non-empty set X (i.e., 'a R b' is never true) is not an equivalence relation, because although it is vacuously symmetric and transitive, but it is not reflexive (except when X is also empty).

**6.** Given an arbitrary equivalence relation R in a set X, R divides X into mutually disjoint subsets $S_i$ called partitions or subdivisions of X provided:

a. All elements of S, are related to each other for all i.

b. No element of Si is related to any element of St if i ≠ j.

c. $\bigcup_{i=1}^{n} S_j = X$ and $S_i \cap S_j = \phi$ if $i \neq j$.

The subsets St are called equivalence classes.

**7. Union, Intersection and Inverse of Equivalence Relations**

a. If R and S are two equivalence relations on a set A, R ∩ S is also an equivalence relation on A.

b. The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

c. The inverse of an equivalence relation is an equivalence relation.

## Top Concepts in Functions

**1. Introduction to functions**

A function from a non-empty set A to another non-empty set B is a correspondence or a rule which associates every element of A to a unique element of B written as $f : A \to B$ such that $f(x) = y$ for all $x \in A$, $y \in B$.

All functions are relations, but the converse is not true.

**2. Domain, Co-domain and Range of a Function**

- If $f : A \to B$ is a function, then set A is the domain, set B is the co-domain and set {f(x) : x ∈ A) is the range of f.
- The range is a subset of the co-domain.
- A function can also be regarded as a machine which gives a unique output in set B corresponding to each input from set A.
- If A and B are two sets having m and n elements, respectively, then the total number of functions from A to B is $n^m$.

**3. Real Function**

- A function $f : A \to B$ is called a real-valued function if B is a subset of R.
- If A and B both are subsets of R, then 'f' is called a real function.
- While describing real functions using mathematical formula, x (the input) is the independent variable and y (the output) is the dependent variable.
- The graph of a real function 'f' consists of points whose co-ordinates (x, y) satisfy $y = f(x)$, for all $x \in$ Domain(f).

**4. Vertical line test**

A curve in a plane represents the graph of a real function if and only if no vertical line intersects it more than once.

**5. One-one Function**

- A function $f : A \to B$ is one-to-one if for all $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$ or $x \neq y \Rightarrow f(x) \neq f(y)$.
- A one-one function is known as an injection or injective function. Otherwise, f is called many-one.

**6. Onto Function**

- A function $f : A \to B$ is an onto function, if for each $b \in B$, there is at least one $a \in A$ such that $f(a) = b$, i.e., if every element in B is the image of some element in A, then f is an onto or surjective function.
- For an onto function, range = co-domain.
- A function which is both one-one and onto is called a bijective function or a bijection.
- A one-one function defined from a finite set to itself is always onto, but if the set is infinite, then it is not the case.

**7. Let A and B be two finite sets and $f : A \to B$ be a function.**

- If f is an injection, then $n(A) \leq n(B)$.
- If f is a surjection, then $n(A) \geq n(B)$.
- If f is a bijection, then $n(A) = n(B)$.

**8.** If A and B are two non-empty finite sets containing m and n elements, respectively, then Number of functions from A to B = $n^m$.

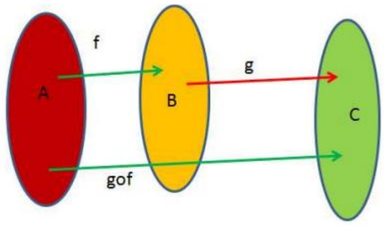
- Number of one-one function from A to B $= \begin{cases} {}^nC_m \times m!, & \text{if } n \geq m \\ 0, & \text{if } n<m \end{cases}$

- Number of onto functions from A to B $= \begin{cases} \sum_{r=1}^{n} (-1)^{n-r} \ {}^{n}C_r r^m, \text{ if } m \geq n \\ 0, \text{ if } m<n \end{cases}$
- Number of one-one and onto functions from A to B $= \begin{cases} n!, \text{ if } m=n \\ 0, \text{ if } m \neq n \end{cases}$

9. If a function f : A → B is not an onto function, then f : A → f(A) is always an onto function.

10. **Composition of Functions**

Let f : A → B and g : B → C be two functions. The composition of f and g, denoted by g o f, is defined as the function g o f: A → C and is given by g o f(x): A → C defined by g o f(x) = g(f(x)) ∀ $x$ ∈ A.




- Composition of f and g is written as g o f and not f o g.
- g o f is defined if the range of f ⊆ domain of g, and f o g is defined if the range of g ⊆ domain of f.
- Composition of functions is not commutative in general i.e., f o g(x) ≠ g o f(x).
- Composition is associative i.e., if f : X → Y, g : Y → Z and h : Z → S are functions, then h o (g o f) = (h o g) o f.
- The composition of two bijections is a bijection.

11. **Inverse of a Function**

- Let f : A → B is a bijection, then g : B → A is inverse of f if f(x) = y ⟺ g(y) = x OR g o f = $I_A$ and f o g = $I_B$
- If g o f = $I_A$ and f is an injection, then g is a surjection.
- If f o g' $I_B$ and f is a surjection, then g is an injection.

12. Let f : A → B and g: B → C be two functions. Then

- g o f: A → C is onto ⇒ g: B → C is onto.
- g o f: A → C is one-one ⇒ f:A → B is one-one.
- g o f: A → C is onto and g: B → C is one-one ⇒ f:A → B is onto.
- g o f: A → C is one-one and f:A → B is onto ⇒ g: B → C is one-one.

13. **Invertible Function**

- A function f : X → Y is defined to be invertible if there exists a function g : Y → X such that gof - $I_x$ and fog = $I_y$.
- The function g is called the inverse of f and is denoted by $f^{-1}$. If f is invertible, then f must be one-one and onto, and conversely, if f is one-one and onto, then f must be invertible.
- If f : A → B and g : B → C are one-one and onto, then g o f : A → C is also one-one and onto. But if g o f is one-one, then only f is one-one and g may or may not be one-one. If g o f is onto, then g is onto and f may or may not be onto.
- Let f : X → Y and g : Y → Z be two invertible functions. Then g o f is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If f: R → R is invertible, f(x) = y, then $f^{-1}(y) = x$ and $(f^{-1})^{-1}$ is the function f itself.

**Binary Operations**

1. A binary operation * on a set A is a function from A × A to A.
2. If * is a binary operation on a set S, then S is closed with respect to *.
3. **Binary operations on R**

- Addition, subtraction and multiplication are binary operations on R, which is the set of real numbers.
- Division is not binary on R; however, division is a binary operation on R - {0} which is the set of non-zero real numbers.

**4. Laws of Binary Operations**

- A binary operation * on the set X is called commutative, if a * b = b * a, for every a, b ∈ X.
- A binary operation * on the set X is called associative, if a (b*c) = (a*b)*c, for every a, b, c ∈ X.
- An element e ∈ A is called an identity of A with respect to * if for each a ∈ A, a * e = a = e * a.
- The identity element of (A, *) if it exists, is unique.

**5. Existence of Inverse**

Given a binary operation * from A × A → A with the identity element e in A, an element a e A is said to be invertible with respect to the operation *, if there exists an element b in A such that a * b = e = b * a and b is called the inverse of a and is denoted by $a^{-1}$.

**6.** If the operation table is symmetric about the diagonal line, then the operation is commutative.

| * | a | b | c | d |
|---|---|---|---|---|
| a | a | b | c | d |
| b | b | c | d | a |
| c | c | d | a | b |
| d | d | a | b | c |

**The operation * is commutative.**

**7. Binary Operation on Natural Numbers**

Addition '+' and multiplication '-' on N, the set of natural numbers, are binary operations. However, subtraction '—' and division are not, because (4, 5) = 4 - 5 = -1 ∈ N and 4/5 = .8 ∈ N.

**8. Number of Binary Operations**

- Let S be a finite set consisting of n elements. Then S x S has $n^2$ elements.
- The total number of functions from a finite set A to a finite set B is $[n(B)]^{n(A)}$. Therefore, total number of binary operations on S is $n^{n^2}$.
- The total number of commutative binary operations on a set consisting of n elements is n $\frac{n(n-1)}{2}$.

**Class : 12th Maths**
**Chapter- 1 : Relations and Functions**

## Relations and Functions

### Composition of functions and Invertible functions

**Invertible functions**

A function $f : x \to y$ is invertible, if $\exists$ a function $g : y \to x$ such that $gof = I_x$ and $fog = I_y$. Then, $g$ is the inverse of $f$. If $f$ is invertible, then it is both one-one and onto and vice-versa. For eg. If $f(x) = x$ and $f : N \to N$, then $f$ is invertible.
**Theorem 1 :** If $f : x \to y$, $g : y \to z$ and $h : z \to s$ are functions, then $ho(got) = (hog)of$.
**Theorem 2 :** Let $f : x \to y$ and $g : y \to z$ be two invertible functions, then $gof$ is invertible and $(gof)^{-1} = f^{-1} o g^{-1}$.

**Composition of functions**

The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by $gof$, and is defined as $gof : A \to C$ given by $gof(x) = g(f(x))$ $\forall x \in A$. e.g. let $A = N$ and $f, g : N \to N$ such that $f(x) = x^2$ and $g(x) = x^3$ $\forall$ $x \in N$. Then $gof(2) = g(f(2)) = g(2^2) = 4^3 = 64$.

### Binary Operations Definition and its types

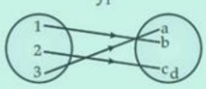
A binary operation '*' on a set A is a function $* : A \times A \to A$ denoted by $a * b$ i.e. $\forall$ $a,b, \in A$, $a * b \in A$. Commutative if $a * b = b * a$ $\forall$ $a,b \in A$. Associative if $(a * b) * c = a * (b * c)$ $\forall a,b, \in A$. $e \in A$ is identity if $a * e = a = e * a$ $\forall a \in A$. and $b \in A$ is the inverse of $a \in A$, if. $a * b = e = b * a$. Addition is a binary operation on the set of integers.

### Types of functions

**One - one (injective)**

$f : x \to y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\forall$ $x_1, x_2 \in x$. Other wise, $f$ is many- one, $f_1$ is one-one.

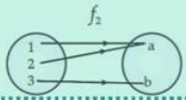
$f_1$




**Onto (Surjective)**

$f : x \to y$ is onto it for energy $y \in Y, \exists$ $x \in X$ S.t. $f(x) = y$, $f_2$ is onto.

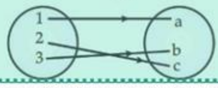
$f_2$




**Bijective**

$f : x \to y$ is both one-one and onto, then $f$ is bijective. $f_3$ is bijective.

$f_3$




### Types of Relations

**Empty Relations**
**Universal relation**

A relation $R : A \to A$ is empty if $a \not R b$ $\forall$ $a, b \in A$. $R = \phi \subset A \times A$.
For $eg$ : $R = \{(a,b) : a = b^2\}$, $A = \{1,5,10\}$

A relation $R : A \to A$ is universal if $a R b$ $\forall$, $a,b \in A$, $R = A \times A$.
if $R = \phi$, then R is universal.

} Trivial Relations

**Reflexive relation**

A relation $R : A \to A$ is reflexive if $aRa$ $\forall$ $a \in A$

**Symmetric Relation**

A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa$ $\forall$ $a, b \in A$

**Transitive relation**

A relation $R : A \times A$ is transitive if $aRb$, $bRc \Rightarrow aRc$ $\forall$ $a,b,c \in A$.

**Equivalence relation**

(reflexive, symmetric, transitive e.g., Let $T$ = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2)\} : T_1$ is congruent to $T_2\}$. Then, $R$ is equivalence.

# Important Questions

## Multiple Choice Questions

1. Let R be the relation in the set {1, 2, 3, 4}, given by:
   R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}.
   Then:
   (a) R is reflexive and symmetric but not transitive
   (b) R is reflexive and transitive but not symmetric
   (c) R is symmetric and transitive but not reflexive
   (d) R is an equivalence relation.
2. Let R be the relation in the set N given by: R = {(a, b): a = b – 2, b > 6}. Then:
   (a) (2, 4) ∈ R
   (b) (3, 8) ∈ R
   (c) (6, 8) ∈ R
   (d) (8, 7) ∈ R.
3. Let A = {1, 2, 3}. Then number of relations containing {1, 2} and {1, 3}, which are reflexive and symmetric but not transitive is:
   (a) 1
   (b) 2
   (c) 3
   (d) 4
4. Let A = (1, 2, 3). Then the number of equivalence relations containing (1, 2) is
   (a) 1
   (b) 2
   (c) 3
   (d) 4.
5. Let f: R → R be defined as $f(x) = x^4$. Then
   (a) f is one-one onto
   (b) f is many-one onto
   (c) f is one-one but not onto
   (d) f is neither one-one nor onto.
6. Let f: R → R be defined as f(x) = 3x. Then
   (a) f is one-one onto
   (b) f is many-one onto
   (c) f is one-one but not onto
   (d) f is neither one-one nor onto.
7. If f: R → R be given by $f(x) = (3 - x^3)^{1/3}$, then fof (x) is
   (a) $x^{1/3}$
   (b) $x^3$
   (c) x
   (d) $3 - x^3$.
8. Let f: R – {$-\frac{4}{3}$} → R be a function defined as: $f(x) = \frac{4x}{3x+4}$, $x \neq -\frac{4}{3}$. The inverse of f is map g: Range f → R -{$-\frac{4}{3}$} given by
   (a) $g(y) = \frac{3y}{3-4y}$
   (b) $g(y) = \frac{4y}{4-3y}$
   (c) $g(y) = \frac{4y}{3-4y}$
   (d) $g(y) = \frac{3y}{4-3y}$
9. Let R be a relation on the set N of natural numbers defined by nRm if n divides m. Then R is
   (a) Reflexive and symmetric
   (b) Transitive and symmetric
   (c) Equivalence
   (d) Reflexive, transitive but not symmetric.
10. Set A has 3 elements, and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is:
    (a) 144
    (b) 12
    (c) 24
    (d) 64

## Very Short Questions:

1. If R = {(x, y) : x + 2y = 8} is a relation in N, write the range of R.
2. Show that a one-one function:
   f {1, 2, 3} → {1, 2, 3} must be onto. (N.C.E.R.T.)
3. What is the range of the function $f(x) = \frac{|x-1|}{x-1}$?
4. Show that the function f : N → N given by f(x) = 2x is one-one but not onto.
5. If f : R → R is defined by f(x) = 3x + 2 find f(f(x)).
6. If $f(x) = \frac{x}{x-1}$, x ≠1 then find fof.
7. If f: R → R is defined by $f(x) = (3 - x^3)^{1/3}$, find fof (x)
8. Are f and q both necessarily onto, if gof is onto?

## Short Questions:

1. Let A be the set of all students of a Boys' school. Show that the relation R in A given by:

   R = {(a, b): a is sister of b} is an empty relation and the relation R' given by :

   R' = {(a, b) : the difference between heights of a and b is less than 3 metres} is an universal relation.

2. Let f : X → Y be a function. Define a relation R in X given by :

   R = {(a,b):f(a) = f(b)}.

   Examine, if R is an equivalence relation.

3. Let R be the relation in the set Z of integers given by:

   R = {(a, b): 2 divides a – b}.

   Show that the relation R is transitive. Write the equivalence class [0].

   Show that the function:

   f : N → N

   given by f(1) = f(2) = 1 and f(x) = x -1, for every x > 2 is onto but not one-one.

4. Find gof and fog, if:

   f : R → R and g : R → R are given by f (x) = cos x and g (x) = $3x^2$. Show that gof ≠ fog.

5. If f(x) = $\frac{4x+3}{6x-4}$, x ≠ $\frac{2}{3}$ find fof(x)

6. Let A = N x N be the set of ail ordered pairs of natural numbers and R be the relation on the set A defined by (a, b) R (c, d) if ad = bc. Show that R is an equivalence relation.

7. Let f: R → R be the Signum function defined as:

   $$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

   and g : R → R be the Greatest Integer Function given by g (x) = [x], where [x] is greatest integer less than or equal to x. Then does fog and of coincide in (0,1]?

## Long Questions:

1. Show that the relation R on R defined as R = {(a, b):a ≤ b}, is reflexive and transitive but not symmetric.

2. Prove that function f : N → N, defined by f(x) = $x^2$ + x + 1 is one-one but not onto. Find inverse of f : N → S, where S is range of f.

3. Let A = {x ∈Z : 0 ≤ x ≤ 12}.

   Show that R = {(a, b) : a, b ∈ A; |a – b| is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. (C.B.S.E 2018)

4. Prove that the function f: [0, ∞) → R given by $f(x) = 9x^2 + 6x - 5$ is not invertible. Modify the co-domain of the function f to make it invertible, and hence find f-1. (C.B.S.E. Sample Paper 2018-19

## Assertion and Reason Questions-

1. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

   a) Both A and R are true and R is the correct explanation of A.

   b) Both A and R are true but R is not the correct explanation of A.

   c) A is true but R is false.

   d) A is false and R is also false.

   **Assertion(A):** Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L1, L2): L1 is perpendicular to L2}.R is not equivalence realtion.

   **Reason (R):** R is symmetric but neither reflexive nor transitive

2. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

   a) Both A and R are true and R is the correct explanation of A.

   b) Both A and R are true but R is not the correct explanation of A.

   c) A is true but R is false.

   d) A is false and R is also false.

   **Assertion (A):** = {(T1, T2): T1 is congruent to T2}. Then R is an equivalence relation.

   **Reason(R):** Any relation R is an equivalence relation, if it is reflexive, symmetric and transitive.

## Case Study Questions-

1. Consider the mapping f: A → B is defined by f(x) = x - 1 such that f is a bijection.

Based on the above information, answer the following questions.

(i) Domain of f is:

(a) R - {2}

(b) R

(c) R - {1, 2}

(d) R - {0}

(ii) Range of f is:

(a) R

(b) R - {2}

(c) R - {0}

(d) R - {1, 2}

(iii) If g: R - {2} → R - {1} is defined by g(x) = 2f(x) - 1, then g(x) in terms of x is:

(a) $\frac{x+2}{x}$

(b) $\frac{x+1}{x-2}$

(c) $\frac{x-2}{x}$

(d) $\frac{x}{x-2}$

(iv) The function g defined above, is:

(a) One-one

(b) Many-one

(c) into

(d) None of these

(v) A function f(x) is said to be one-one if.

(a) $f(x_1) = f(x_2) \Rightarrow -x_1 = x_2$

(b) $f(-x_1) = f(-x_2) \Rightarrow -x_1 = x_2$

(c) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

(d) None of these

**2.** A relation R on a set A is said to be an equivalence relation on A if it is:

I. Reflexive i.e., (a, a) ∈ R ∀ a ∈ A.

II. Symmetric i.e., (a, b) ∈ R ⇒ (b, a) ∈ R ∀ a, b ∈ A.

III. Transitive i.e., (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R ∀ a, b, c ∈ A.

Based on the above information, answer the following questions.

(i) If the relation R = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} defined on the set A = {1, 2, 3}, then R is:

a) Reflexive

b) Symmetric

c) Transitive

d) Equivalence

(ii) If the relation R = {(1, 2), (2, 1), (1, 3), (3, 1)} defined on the set A = {1, 2, 3}, then R is:

a) Reflexive

b) Symmetric

c) Transitive

d) Equivalence

(iii) If the relation R on the set N of all natural numbers defined as R = {(x, y): y = x + 5 and x < 4}, then R is:

a) Reflexive

b) Symmetric

c) Transitive

d) Equivalence

(iv) If the relation R on the set A = {1, 2, 3, ........, 13, 14} defined as R = {(x, y): 3x - y = 0}, then R is:

a) Reflexive

b) Symmetric

c) Transitive

d) Equivalence

(v) If the relation R on the set A = {I, 2, 3} defined as R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}, then R is:

a) Reflexive only

b) Symmetric only

c) Transitive only

d) Equivalence

# Answer Key

## Multiple Choice Questions

1. (b) R is reflexive and transitive but not symmetric
2. (c) $(6, 8) \in R$
3. (a) 1
4. (b) 2
5. (c) f is neither one-one nor onto.
6. f is one-one onto
7. x
8. $g(y) = \frac{4y}{4-3y}$
9. (b) Transitive and symmetric
10. 24

## Very Short Answer:

1. **Solution:** Range of R = {1, 2, 3}.

   [∵ When x = 2, then y = 3, when x = 4, then y = 2, when x = 6, then y = 1]

2. **Solution:** Since 'f' is one-one,

   ∴ under 'f', all the three elements of {1, 2, 3} should correspond to three different elements of the co-domain {1, 2, 3}.

   Hence, 'f' is onto.

3. **Solution:** When x > 1,

   than $f(x) = \frac{x-1}{x-1} = 1$.

   When x < 1,

   than $f(x) = \frac{-(x-1)}{x-1} = -1$

   Hence, Rf = {-1, 1}.

4. **Solution:**

   Let $x_1, x_2 \in N$.

   Now, $f(x_1) = f(x_2)$

   $\Rightarrow 2x_1 = 2x_2$

   $\Rightarrow x_1 = x_2$

   ⇒ f is one-one.

   Now, f is not onto.

   ∵ For $1 \in N$, there does not exist any $x \in N$ such that f(x) = 2x = 1.

   Hence, f is ono-one but not onto.

5. **Solution:**

   f(f(x)) = 3 f(x) + 2

   = 3(3x + 2) + 2 = 9x + 8.

6. **Solution:**

   $$fof(x) = f(f(x)) = \frac{f(x)}{f(x)-1}$$

   $$= \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-x+1}$$

   $$= \frac{x}{1} = x.$$

7. **Solution:**

   $fof(x) = f(f(x)) = (3-(f(x))^3)^{1/3}$

   $= (3 - ((3 - x^3)^{1/3})^3)^{1/3}$

   $= (3 - (3 - x^3))^{1/3} = (x^3)^{1/3} = x$.

8. **Solution:**

   Consider f: {1, 2, 3, 4} → {1, 2, 3, 4}

   and g: {1, 2, 3,4} → {1,2.3} defined by:

   f(1) = 1, f(2) = 2, f(3) = f(4) = 3

   g (1) = 1, g (2) = 2, g (3) = g (4) = 3.

   ∴ gof = g (f(x)) {1, 2,3}, which is onto

   But f is not onto.

   [∵ 4 is not the image of any element]

## Short Answer:

1. **Solution:**

   (i) Here R = {(a, b): a is sister of b}.

   Since the school is a Boys' school,

   ∴ no student of the school can be the sister of any student of the school.

   Thus R = Φ Hence, R is an empty relation.

   (ii) Here R' = {(a,b): the difference between heights of a and b is less than 3 metres}.

   Since the difference between heights of any two students of the school is to be less than 3 metres,

   ∴ R' = A x A. Hence, R' is a universal relation.

2. **Solution:**

   For each $a \in X$, $(a, a) \in R$.

   Thus, R is reflexive. [∵ f (a) = f(a)]

   Now $(a, b) \in R$

   ⇒ f(a) = f(b)

   ⇒ f(b) = f (a)

⇒ (b, a) ∈ R.

Thus R is symmetric.

And (a, b) ∈ R

and (b, c) ∈ R

⇒ f(a) = f(b)

and f(b) = f(c)

⇒ f(a)= f(c)

⇒ (a, c) ∈ R.

Thus R is transitive.

Hence, R is an equivalence relation.

**3. Solution:**

Let 2 divide (a – b) and 2 divide (b – c), where a,b,c ∈ Z

⇒ 2 divides [(a – b) + (b – c)]

⇒ 2 divides (a – c).

Hence, R is transitive.

And [0] = {0, ± 2, ± 4, ± 6,...].

**4. Solution:**

Since f(1) = f(2) = 1,

∴ f(1) = f(2), where 1 ≠ 2.

∴ 'f' is not one-one.

Let y ∈ N, y ≠ 1,

we can choose x as y + 1 such that f(x) = x – 1

= y + 1 – 1 = y.

Also 1 ∈ N, f(1) = 1.

Thus 'f ' is onto.

Hence, 'f ' is onto but not one-one.

**5. Solution:**

We have:

f(x) = cos x and g(x) = $3x^2$.

∴ gof (x) = g (f(x)) = g (cos x)

= 3 $(\cos x)^2$ = 3 $\cos^2 x$

and fog (x) = f(g (x)) = f($3x^2$) = cos $3x^2$.

Hence, gof ≠ fog.

**6. Solution:**

We have: $\frac{4x+3}{6x-4}$ ...(1)

∴ fof(x) - f (f (x))

$$=\frac{4f(x)+3}{6f(x)-4}$$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} \quad \text{[Using (1)]}$$

$$=\frac{16x+12+18x-12}{24x+18-24x+16}$$

$$=\frac{34x}{34}=x.$$

**7. Solution:**

Given: (a, b) R (c, d) if and only if ad = bc.

(I) (a, b) R (a, b) iff ab – ba, which is true.

[∵ ab = ba ∀ a, b ∈ N]

Thus, R is reflexive.

(II) (a, b) R (c,d) ⇒ ad = bc

(c, d) R (a, b) ⇒ cb = da.

But cb = be and da = ad in N.

∴ (a, b) R (c, d) ⇒ (c, d) R (a, b).

Thus, R is symmetric.

(III) (a,b) R (c, d)

⇒ ad = bc ...(1)

(c, d) R (e,f)

⇒ cf = de ... (2)

Multiplying (1) and (2), (ad). (cf) – (be), (de)

⇒ af = be

⇒ (a,b) = R(e,f).

Thus, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

**8. Solution:**

For x ∈ (0,1].

$$(fog)(x)=f(g(x))=f([x])$$

$$=\begin{cases} f(0); & \text{if } 0<x<1 \\ f(1); & \text{if } x=1 \end{cases}$$

$$\Rightarrow f(g(x))=\begin{cases} 0; & \text{if } 0<x<1 \\ 1; & \text{if } x=1 \end{cases} \quad \text{....(1)}$$

And (gof) (x) = g(f(x)) = g(1)

[∵ f(x) = 1 ∀ x > 0]

= [1] = 1

⇒ (gof) (x) = 1 ∀ x ∈ (0, 1] ...(2)

From (1) and (2), (fog) and (gof) do not coincide in (0, 1].

## Long Answer:

**1. Solution:**

We have: R = {(a, b)} = a ≤ b}.

Since, a ≤ a ∀ a ∈ R,

∴ (a, a) ∈ R,

Thus, R reflexive.

Now, (a, b) ∈ R and (b, c) ∈ R

⇒ a ≤ b and b ≤ c

⇒ a ≤ c

⇒ (a, c) ∈ R.

Thus, R is transitive.

But R is not symmetric

[∵ (3, 5) ∈ R but (5, 3) ∉ R as 3 ≤ 5 but 5 > 3]

**Solution:**

Let $x_1, x_2 \in N$.

Now, $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^2+x_1+1=x_2^2+x_2+1$

$\Rightarrow \quad x_1^2+x_1=x_2^2+x_2$

$\Rightarrow \quad (x_1^2-x_2^2)+(x_1+x_2)=0$

$\Rightarrow \quad (x_1+x_2)+(x_1+x_2+1)=0$

$\Rightarrow \quad x_1-x_2=0 \qquad [\because x_1+x_2+1\neq 0]$

$\Rightarrow \quad x_1 = x_2$.

Thus, f is one-one.

Let $y \in N$, then for any $x$,

$f(x) = y$ if $y = x^2 + x + 1$

$\Rightarrow \quad y=\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}$

$\Rightarrow \quad y=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

$\Rightarrow \quad x+\frac{1}{2}=\pm\sqrt{y-\frac{3}{4}}$

$\Rightarrow \quad x=\pm\frac{\sqrt{4y-3}}{2}-\frac{1}{2}$

$\Rightarrow \quad x=\frac{\pm\sqrt{4y-3}-1}{2}$

$\left[\frac{-\sqrt{4y-3}-1}{2}\notin N \text{ for any value of } y\right]$

Now, for $y=\frac{3}{4}, x=-\frac{1}{2}\notin N.$

Thus, $f$ is not onto.

⇒ $f(x)$ is not invertible.

Since, x > 0, therefore, $\frac{\sqrt{4y-3}-1}{2}>0$

$\Rightarrow \quad \sqrt{4y-3}>1$

$\Rightarrow \quad 4y - 3 > 1$

$\Rightarrow \quad 4y > 4$

$\Rightarrow \quad y > 1.$

Redefining, $f : (0, \infty) \to (1, \infty)$ makes

$f(x) = x^2 + x + 1$ on onto function.

Thus, $f(x)$ is bijection, hence $f$ is invertible and $f^1$ : $(1, \infty) \to (1, 0)$

$$f^1(y)=\frac{\sqrt{4y-3}-1}{2}$$

**2. Solution:**

We have:

R = {(a, b): a, b ∈ A; |a – b| is divisible by 4}.

(i) Reflexive: For any a ∈ A,

∴ (a, b) ∈ R.

|a – a| = 0, which is divisible by 4.

Thus, R is reflexive.

**Symmetric:**

Let (a, b) ∈ R

⇒ |a – b| is divisible by 4

⇒ |b – a| is divisible by 4

Thus, R is symmetric.

**Transitive:** Let (a, b) ∈ R and (b, c) ∈ R

⇒ |a – b| is divisible by 4 and |b – c| is divisible by 4

⇒ |a – b| = 4λ

⇒ a – b = ±4λ .............(1)

and |b – c| = 4μ

⇒ b – c = ± 4μ .............(2)

Adding (1) and (2),

(a-b) + (b-c) = ±4(λ + μ)

⇒ a – c = ± 4 (λ + μ)

⇒ (a, c) ∈ R.

Thus, R is transitive.

Now, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

(ii) Let 'x' be an element of A such that (x, 1) ∈ R

⇒ |x – 1| is divisible by 4

⇒ x – 1 = 0,4, 8, 12,...

⇒ x = 1, 5, 9, 13, ...

Hence, the set of all elements of A which are related to 1 is {1, 5, 9}.

(iii) Let $(x, 2) \in R$.

Thus |x – 2| = 4k, where k ≤ 3.

∴ x = 2, 6, 10.

Hence, equivalence class [2] = {2, 6, 10}.

**3. Solution:**

Let y ∈ R.

For any x, f(x) = y if y = $9x^2 + 6x - 5$

⇒ y = $(9x^2 + 6x + 1) - 6$

= $(3x + 1)^2 - 6$

⇒ $3x+1=\pm\sqrt{y+6}$

⇒ $x=\dfrac{\pm\sqrt{y+6}-1}{3}$

⇒ $x=\dfrac{\sqrt{y+6}-1}{3}$

$\left[\because \dfrac{-\sqrt{y+6}-1}{3} \notin [0,\infty)\ \textit{for any value of}\ y\right]$

For y = –6 ∈ R, x = $-\dfrac{1}{3} \notin [0,\infty)$.

Thus, $f(x)$ is not invertible.

Since, $x \geq 0$,

∴ $\dfrac{\sqrt{y+6}-1}{3} \geq 0$

⇒ $\sqrt{y+6} \geq 1$

⇒ y + 6 ≥ 1

⇒ y ≥ –5.

We redefine,

$f$: $[0, \infty) \to [-5, \infty)$.

which makes $f(x) = 9x^2 + 6x - 5$ an onto function.

Now, $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

⇒ $(3x_1 + 1)^2 = (3x_2 + 1)^2$

⇒ $[(3x_1 + 1) + (3x_2 + 1)]\ [(3x_1 + 1) - (3x_2 + 1)]$

⇒ $[3(x_1 + x_2) + 2]\ [3(x_1 - x_2)] = 0$

⇒ $x_1 = x_2$

$[\because 3(x_1 + x_2) + 2 > 0]$

Thus, f(x) is one-one.

∴ $f(x)$ is objective, hence f is invertible and $f^{-1}$: $[-5, \infty) \to [0, \infty)$

$$f^{-1}(y)=\frac{\sqrt{y+6}-1}{3}$$

## Assertion and Reason Answers-

**1.** (a) Both A and R are true and R is the correct explanation of A.

**2.** (a) Both A and R are true and R is the correct explanation of A.

## Case Study Answers-

**1. Answer:**

(i) (a) R – {2}

**Solution:**

For f(x) to be defined x - 2; ≠ 0 i.e., x; ≠ 2.

∴ Domain of f = R - {2}

(ii) (b) R – {2}

**Solution:**

Let y = f(x), then $y=\dfrac{x-1}{x-2}$

⇒ $xy - 2y = x - 1 \Rightarrow xy - x = 2y -$

⇒ $x=\dfrac{2y-1}{2y-1}$

Since, x ∈∈ R – {2}, therefore y ≠ 1

Hence, range of f = R - {1}

(iii) (d) $\dfrac{x}{x-2}$

**Solution:**

We have, g(x) = 2f(x) – 1

$=2\left(\dfrac{x-1}{x-2}\right)-1=\dfrac{2x-2-x+2}{x-2}=\dfrac{x}{x-2}$

(iv) (a) One-one

**Solution:**

We have, $g(x)=\dfrac{x}{x-2}$

Let = $g(x_1)=g(x_2) \Rightarrow \dfrac{x_1}{x_1-2}=\dfrac{x_2}{x_2-2}$

⇒ $x_1x_2 - 2x_1 = x_1x_2 - 2x_2 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

Thus, $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

Hence, g(x) is one-one.

(v) (c) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

**2. Answer:**

(i) (a) Reflexive

**Solution:**

Clearly, (1, 1), (2, 2), (3, 3), ∈ R. So, R is reflexive on A.

Since, (1, 2) ∈ R but (2, 1) ∉ R. So, R is not symmetric on A.

Since, (2, 3), ∈ R and (3, 1) ∈ R but (2, 1) ∉ R. So, R is not transitive on A.

(ii) (b) Symmetric

**Solution:**

Since, (1, 1), (2, 2) and (3, 3) are not in R. So, R is not reflexive on A.

Now, (1, 2) ∈ R ⇒ (2, 1) ∈ R and (1, 3) ∈ R ⇒ (3, 1) ∈ R. So, R is symmetric,

Clearly, (1, 2) ∈ R and (2, 1) ∈ R but (1, 1) ∉ R. So, R is not transitive on A.

(iii) (c) Transitive

**Solution:**

We have, R = {(x, y): y = x + 5 and x < 4}, where x, y ∈ N.

∴ R = {(1, 6), (2, 7), (3, 8)}

Clearly, (1, 1), (2, 2) etc. are not in R. So, R is not reflexive.

Since, (1, 6) ∈ R but (6, 1) ∉ R. So, R is not symmetric.

Since, (1, 6) ∈ R and there is no order pair in R which has 6 as the first element.

Same is the case for (2, 7) and (3, 8). So, R is transitive.

(iv) (d) Equivalence

**Solution:**

We have, R = {(x, y): 3x - y = 0}, where x, y ∈ A = {1,2, ......, 14}.

∴ R = {(1, 3), (2, 6), (3, 9), (4, 12)}

Clearly, (1, 1) ∉ R. So, R is not reflexive on A.

Since, (1, 3) ∈ R but (3, 1) ∉ R. So, R is not symmetric on A.

Since, (1, 3) ∈ Rand (3, 9) ∈ R but (1, 9) ∉ R. So, R is not transitive on A.

(v) (d) Equi0076alence

**Solution:**

Clearly, (1, 1), (2, 2), (3, 3) ∈ R. So, R is reflexive on A.

We find that the ordered pairs obtained by interchanging the components of ordered pairs in R are also in R. So, R is symmetric on A. For 1, 2, 3 ∈ A such that (1, 2) and (2, 3) are in R implies that (1, 3) is also, in R. So, R is transitive on A. Thus, R is an equivalence relation.

# Inverse Trigonometric Functions 2

1. The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

| Functions | Domain | Range (Principal Value Branches) |
|---|---|---|
| $y = \sin^{-1}x$ | [−1, 1] | $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ |
| $y = \cos^{-1}x$ | [−1, 1] | $[0, \pi]$ |
| $y = \text{cosec}^{-1}x$ | R − [−1, 1] | $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]_{-\{0\}}$ |
| $y = \sec^{-1}x$ | R − [−1, 1] | $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ |
| $y = \tan^{-1}x$ | R | $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ |
| $y = \cot^{-1}x$ | R | $[0, \pi]$ |

2. $\sin^{-1}x$ should not be confused with ($\sin^{-1}x$). In fact, $(sin\ x)^{-1} = \frac{1}{sinsin\ x}$ And similarly for other trigonometric functions.
3. The value of an inverse trigonometric functions which lies in its principal value branch is called the principal value of that inverse trigonometric functions.
4. **For suitable values of domain, we have**
   - $y = \sin^{-1} x \Rightarrow x = \sin y$
   - $x = \sin y \Rightarrow y = \sin^{-1} x$
   - $\sin(\sin^{-1} x) = x$
   - $\sin^{-1}(\sin x) = x$
   - $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1}x$
   - $\cos^{-1}(-x) = \pi - \cos^{-1} x$
   - $\cos^{-1}\frac{1}{x} = \sec^{-1} x$
   - $\cot^{-1}(-x) = \pi - \cot^{-1} x$
   - $\tan^{-1}\frac{1}{x} = \cot^{-1} x$

- $\operatorname{cosec}^{-1}\frac{1}{x}=\sin^{-1}x$
- $\sec^{-1}(-x)=\pi-\sec^{-1}x$
- $\sec^{-1}\frac{1}{x}=\cos^{-1}x$
- $\sin^{-1}(-x)=-\sin^{-1}x$
- $\tan^{-1}(-x)=-\tan^{-1}x$
- $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$
- $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x$
- $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}$
- $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$
- $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$
- $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$
- $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$
- $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$
- $2\cos^{-1}x=\cos^{-1}(2x^2-1)$
- $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\tan^{-1}\left(\frac{2x}{1-\sqrt{x}}\right)$
- $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$
- $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$
- $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

5. **Conversion:**

- $\sin^{-1}x=\cos^{-1}\frac{\sqrt{1-x^2}}{x}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\operatorname{cosec}^{-1}\frac{1}{x}$
- $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{\sqrt{1-x^2}}{x}=\cot^{-1}\frac{x}{\sqrt{1-x^2}}=\sec^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$
- $\tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\cos^{-1}\frac{1}{\sqrt{1+x^2}}=\sec^{-1}\sqrt{1+x^2}=\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}=\cot^{-1}\frac{1}{x}$
- $\cot^{-1}x=\sin^{-1}\frac{1}{\sqrt{1+x^2}}=\cos^{-1}\frac{x}{\sqrt{1+x^2}}=\sec^{-1}\frac{1}{x}=\sec^{-1}\frac{\sqrt{1+x^2}}{x}=\operatorname{cosec}^{-1}\sqrt{1+x^2}$
- $\sec^{-1}x=\tan^{-1}\frac{\sqrt{x^2-1}}{1}=\cot^{-1}\frac{1}{\sqrt{x^2-1}}=\sin^{-1}\frac{\sqrt{x^2-1}}{x}=\cos^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

- $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{\sqrt{x^2-1}}=\cot^{-1}\sqrt{x^2-1}=\sec^{-1}\frac{x}{\sqrt{x^2-1}}=\cos^{-1}\frac{\sqrt{x^2-1}}{x}$

6. **Some other properties of Inverse Trigonometric Function:**

- $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$
- $\cot^{-1}\frac{x}{\sqrt{a^2-x^2}}=\cos^{-1}\frac{x}{a}$
- $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\operatorname{cosec}^{-1}\frac{x}{a}$
- $\cot^{-1}\frac{x}{\sqrt{a^2-x^2}}=\sec^{-1}\frac{x}{a}$

**Class : 12th Maths**
**Chapter- 2 : Inverse Trigonometric Functions**

# Inverse Trigonometric Functions

## Domain and range of inverse trigonometric functions

(i) $y = \sin^{-1}x$. Domain = [–1,1], Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $y = \cos^{-1}x$. Domain = [–1,1] Range = $[0, \pi]$

(iii) $y = \operatorname{cosec}^{-1}x$. Domain = R–(–1,1), Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iv) $y = \sec^{-1}x$. Domain = R–(–1,1), Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(v) $y = \tan^{-1}x$. Domain = R, Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $y = \cot^{-1}x$. Domain = R, Range = $(0, \pi)$.

## Trigonometric functions

(I) $\sin : R \to [-1,1]$

(ii) $\cos : R \to [-1,1]$

(iii) $\tan : R - \left\{x : x = (2n+1)\frac{\pi}{2} in \in Z\right\} \to R$

(iv) $\cot : R - \{x : x = h\pi, n \in Z\} \to R$

(v) $\sec : R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to R - (-1,1)$

(vi) $\operatorname{cosec} : R - \{x : x = h\pi, n \in Z\} \to R - (-1,1)$

## Principal Value branch and principal value

The range of an inverse trigonometric function is the principal value branch and those values which lies in the principal value branch is called the principal value of that inverse trigonometric functions.

## Some important relations

$\sin^{-1}x \neq (\sin x)^{-1}.(\sin x)^{-1} = \frac{1}{\sin x}$ and same for other trigonometric functions.

(i) $y = \sin^{-1}x \Rightarrow x = \sin y$

(ii) $x = \sin y \Rightarrow y = \sin^{-1}x$

(iii) $\sin(\sin^{-1}x) = x$

(iv) $\sin^{-1}(\sin x) = x$

(v) $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x$

(vi) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(vii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x$

(viii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(ix) $\tan^{-1}\frac{1}{x} = \cot^{-1}x$

(x) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(xi) $\sin^{-1}(-x) = -\sin^{-1}x$

(xii) $\tan^{-1}(-x) = -\tan^{-1}x$

(xiii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(xiv) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

(xv) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

(xvi) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

(xvii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

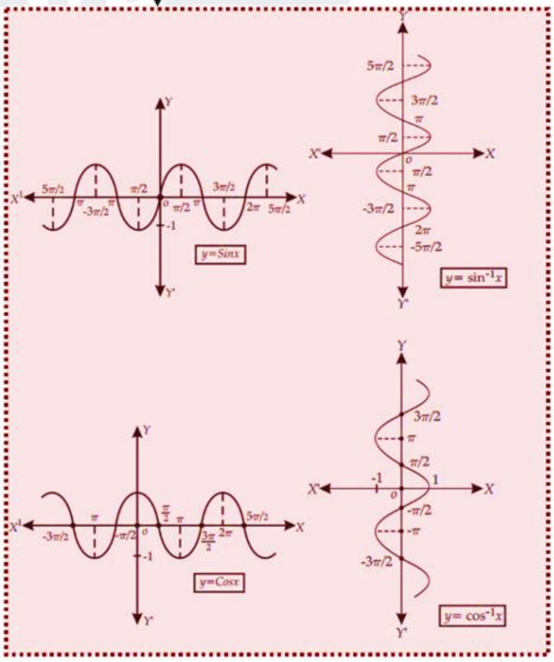
(xviii) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1+x^2}{1-x^2}$

For *eg* : to find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$,

$\Rightarrow \sin y = \frac{1}{\sqrt{2}}$ The range or the principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ So, the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$

## Graphs of trigonometric functions and inverse trigonometric functions



# Important Questions

## Multiple Choice questions-

1. If $\sin^{-1}x+\sin^{-1}y=\frac{2x}{3}$, then the value of $\cos^{-1}x$ + $\text{cost}^{-1}y$ is

(a) $\frac{2\pi}{3}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(δ) $\pi$

2. $\tan^{-1}(\sqrt{3})-\sec^{-1}(-2)$ is equal to:

(ε) $\pi$

(a) $-\frac{\pi}{3}$

(b) $\frac{\pi}{3}$

(c) $\frac{2\pi}{3}$

3. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to:

(a) $\frac{7\pi}{6}$

(b) $-\frac{5\pi}{6}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{6}$

4. $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to:

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$

(d) 1

5. $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$ is equal to:

(α) $\pi$

(b) $-\frac{\pi}{2}$

(c) 0

(d) $2\sqrt{3}$

6. $\sin(\tan^{-1}x), |x|<1,$ is equal to:

(a) $\frac{x}{\sqrt{1-x^2}}$

(b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{x}{\sqrt{1+x^2}}$

(d) $\frac{x}{\sqrt{1+x^2}}$

7. $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, then $x$ is equal to:

(a) $0, \frac{1}{2}$

(b) $1, \frac{1}{2}$

(c) 0

(d) $\frac{1}{2}$

8. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$ is equal to:

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$

(d) $-\frac{3\pi}{4}$

9. The value of $\sin^{-1}\left(\cos\left(\frac{43\pi}{5}\right)\right)$ is

(a) $\frac{3\pi}{5}$

(b) $\frac{-7\pi}{5}$

(c) $\frac{\pi}{10}$

(d) $-\frac{-\pi}{10}$

10. The principal value of the expression $\cos^{-1}[\cos(-680°)]$ is

(a) $\frac{2\pi}{9}$

(b) $\frac{-2\pi}{9}$

(c) $\frac{34\pi}{9}$

(d) $-\frac{\pi}{9}$

## Very Short Questions:

1. Find the principal value of sin-1 ( $\frac{1}{2}$ )
2. What is the principal value of:
   cos-1 (cos $\frac{2\pi}{3}$ + sin-1 (sin $\frac{2\pi}{3}$)?
3. Find the principal value of:
   $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$.
4. Evaluate: $\tan^{-1}(2\cos(2\sin^{-1}(\frac{1}{2})))$
5. Find the value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$.
6. If $\sin^{-1}(\frac{1}{3}) + \cos^{-1} x = \frac{\pi}{2}$, then find x.
7. If $\sec^{-1}(2) + \text{cosec}^{-1}(y) = \frac{\pi}{2}$, then find y.
8. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}(-\frac{1}{2})\right]$
9. Prove the following:
   $$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}.$$
10. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$, xy < 1, then write the value of the x + y + xy.

## Short Questions:

1. Express $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$
   Where $-\frac{\pi}{4} < x < \frac{\pi}{4}$, in the simples form.
2. Prove that:
   $$\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$$
3. Prove that:
   $$\sin^{-1}\frac{8}{17}+\cos^{-1}\frac{4}{5}=\cos^{-1}\frac{36}{77}$$
4. Solve the following equation:
   $$\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$$
5. Solve the following equation:
   $$2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x),\ x\neq\frac{\pi}{2}$$
6. Solve the following equation
   $$\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$
7. Prove that
   $$3\cos^{-1}x=\cos^{-1}(4x^3-3x),\ x\in\left[\frac{1}{2},1\right]$$

## Long Questions:

1. prove that
   $$\frac{1}{2}\le x\le 1, \text{ then } \cos^{-1}x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]=\frac{\pi}{3}$$
2. Find the value of:
   $$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$
3. Prove that:
   $$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$$
4. $2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}.$

## Assertion and Reason Questions:

**1.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

e) Both A and R are false.

**Assertion(A):** A relation R = {(1, 1), (1, 3), (3, 1), (3, 3), (3, 5)} defined on the set A = {1,3,5} is reflexive.

**Reason (R):** A relation R on the set A is said to be transitive if for. (a, b) ∈R and (b, c)∈R, we have (a,c)∈R.

**2.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

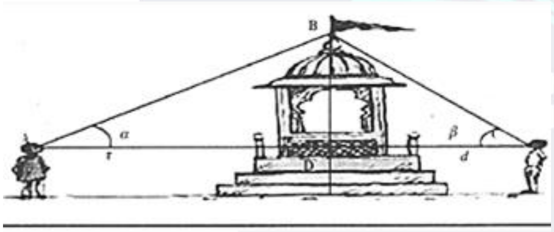
e) Both A and R are false.

**Assertion (A):** A relation R = {(a,b):|a−b| < 2}defined on the set A = {1,2,3,4,5} is reflexive.

**Reason(R):** A relation R on the set A is said to be reflexive if for (a,b) ∈R and(b,c)∈R, we have (a,c)∈R.

## Case Study Questions:

**1.**



Two men on either side of a temple of 30 meters high observe its top at the angles of elevation α and β respectively, (as shown in the figure above). The distance between the two men is $40\sqrt{3}$ meters and the distance between the first person A and the temple is $30\sqrt{3}$ meters. Based on the above information answer the following:

(i) $\angle CAB = \alpha =$

(a) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(b) $\sin^{-1}\left(\frac{1}{2}\right)$

(c) $\sin^{-1}(2)$

(d) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(ii) $\angle CAB = \alpha =$

(a) $\cos^{-1}\left(\frac{1}{5}\right)$

(b) $\cos^{-1}\left(\frac{2}{5}\right)$

(c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(d) $\cos^{-1}\left(\frac{4}{5}\right)$

(iii) $\angle BCA = \alpha =$

(a) $\tan^{-1}\left(\frac{1}{2}\right)$

(b) $\tan^{-1}(2)$

(c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(d) $\tan^{-1}\left(\sqrt{3}\right)$

(iv) $\angle ABC =$

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{6}$

(c) $\frac{\pi}{2}$

(d) $\frac{\pi}{3}$

(v) Domain and Range of $\cos^{-1} x$ =

(a) (−1, 1), (0, π)

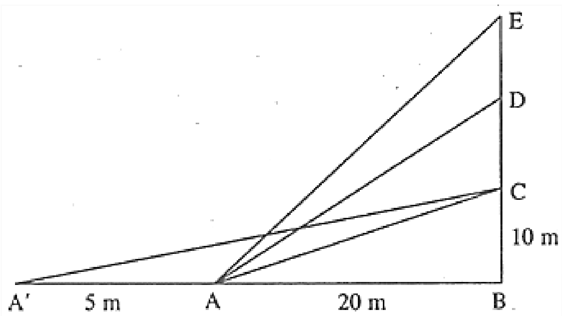
(b) [−1, 1], (0, π)

(c) [−1, 1], [0, π]

(d) $(-1, 1), \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

**2.** The Government of India is planning to fix a hoarding board at the face of a building on the road of a busy market for awareness on COVID-19 protocol. Ram, Robert and Rahim are the three engineers who are working on this project. "A" is considered to be a person viewing the hoarding board 20 metres away from the building, standing at the edge of a pathway nearby. Ram, Robert and Rahim suggested to the firm to place the hoarding board at three different locations namely C, D and E. "C" is at the height of 10 metres from the ground level.

For the viewer A, the angle of elevation of "D" is double the angle of elevation of "C". The angle of elevation of "E" is triple the angle of elevation of "C" for the same viewer. Look at the figure given and based on the above information answer the following:




(i) Measure of $\angle CAB =$

(a) $\tan^{-1}(2)$

(b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $\tan^{-1}(1)$

(d) $\tan^{-1}(3)$

(ii) Measure of $\angle DAB =$

(a) $\tan^{-1}\left(\frac{3}{4}\right)$

(b) $\tan^{-1}(3)$

(c) $\tan^{-1}\left(\frac{4}{3}\right)$

(d) $\tan^{-1}(4)$

(iii) Measure of $\angle EAB =$

(a) $\tan^{-1}(11)$

(b) $\tan^{-1} 3$

(c) $\tan^{-1}\left(\frac{2}{11}\right)$

(d) $\tan^{-1}\left(\frac{11}{2}\right)$

(iv) 'A' is another viewer standing on the same line of observation across the road. If the width of the road is 5 meters, then the difference between $\angle CAB$ and $\angle CA'B$ is

(a) $\tan^{-1}\left(\frac{1}{2}\right)$

(b) $\tan^{-1}\left(\frac{1}{12}\right)$

(c) $\tan^{-1}\left(\frac{2}{5}\right)$

(d) $\tan^{-1}\left(\frac{11}{21}\right)$

(v) Domain and Range of $\tan^{-1} x =$

(a) $R^+, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(b) $R^-, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

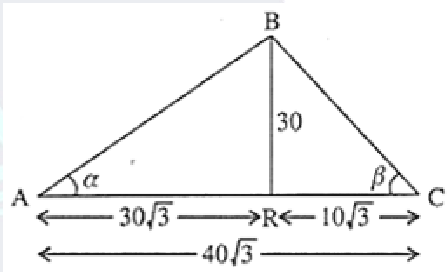
(c) $R, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) $R, \left(0, \frac{\pi}{2}\right)$

## Case Study Answers:

**1.** (i) (b)

**Solution:**



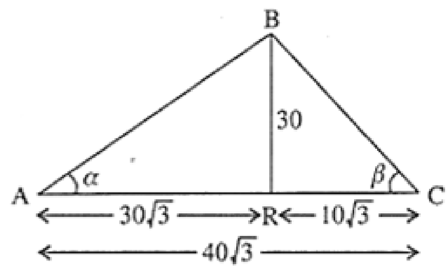

$AB^2 = (AD)^2 + (BD)^2 = (30\sqrt{3})^2 + (30)^2$

$= 2700 + 900 = 3600 \Rightarrow AB = 60\ m$

$\sin\alpha = \frac{BD}{AB} = \frac{30}{60} = \frac{1}{2} \Rightarrow \alpha = \sin^{-1}\left(\frac{1}{2}\right)$

(ii) (c)

**Solution:**



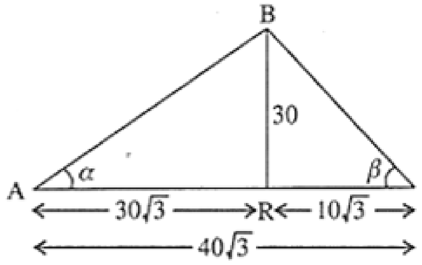

$AB^2 = (AD)^2 + (BD)^2 = (30\sqrt{3})^2 + (30)^2$

$= 2700 + 900 = 3600 \Rightarrow AB = 60\ m$

$\cos\alpha = \frac{AD}{AB} = \frac{30\sqrt{3}}{60} = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii) (d)

**Solution:**



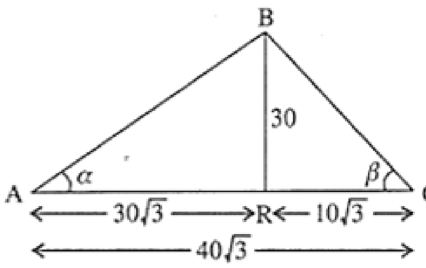

$$AB^2 = (AD)^2 + (BD)^2 = (30\sqrt{3})^2 + (30)^2$$

=2700 + 900 = 3600 $\Rightarrow$ AB = 60 *m*

$$\tan\beta = \frac{BD}{DC} = \frac{30}{10\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \beta = \tan^{-1}(\sqrt{3})$$

(iv) (c)

**Solution:**




$$AB^2 = (AD)^2 + (BD)^2 = (30\sqrt{3})^2 + (30)^2$$

=2700 + 900 = 3600 $\Rightarrow$ AB = 60 *m*

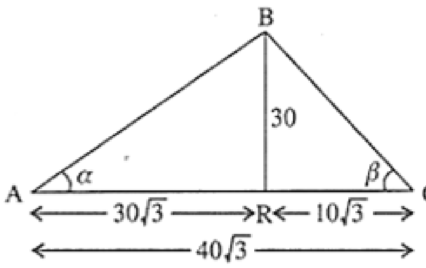
$$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30°; \tan\beta = \sqrt{3}$$

$$\Rightarrow \beta = 60° \therefore \underline{|ABC} + \alpha + \beta = 180°$$

$$\Rightarrow \underline{|ABC} + 30° + 60° = 90° = \frac{\pi}{2}$$

(v) (c)

**Solution:**




$$AB^2 = (AD)^2 + (BD)^2 = (30\sqrt{3})^2 + (30)^2$$

=2700 + 900 = 3600 $\Rightarrow$ AB = 60 *m*

of $\cos^{-1} x = [-1, 1]$ Range of $\cos^{-1} x = [0, \pi]$

**2.**

(i) (b)

**Solution:**

Let $\underline{|BAC} = \alpha$ and $\underline{|BA'C} = \beta \therefore \underline{|BAC} = 2\alpha$

and $\underline{|BAE} = 3\alpha$ In $\Delta BAC \tan\alpha = \frac{10}{20} = \frac{1}{2}$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2}\right)$$

(ii) (c)

**Solution:**

We know that $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}$

$$= \frac{1}{\frac{3}{4}} = \frac{4}{3} \therefore 2\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

(iii) (d)

**Solution:**

Also $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} = \frac{3\times\frac{1}{2}-\left(\frac{1}{2}\right)^3}{1-3\left(\frac{1}{2}\right)^2}$

$$= \frac{\frac{3}{2}-\frac{1}{8}}{1-\frac{3}{4}} = \frac{\frac{11}{8}}{\frac{1}{4}} = \frac{11}{2} \Rightarrow 3\alpha = \tan^{-1}\left(\frac{11}{2}\right)$$

(iv) (b)

**Solution:**

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{1}{2}-\frac{2}{5}}{1+\frac{1}{2}\times\frac{2}{5}}$$

$$= \frac{1}{12}\left\{\because \tan\beta = \frac{BC}{AB} = \frac{10}{25}\right\}$$

(v) (c)

**Solution:**

$$R, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

# Answer Key

## Multiple Choice Questions

1. **Answer:** (b) $\frac{\pi}{3}$
2. **Answer:** (b) $-\frac{\pi}{3}$
3. **Answer:** (b) $-\frac{5\pi}{6}$
4. **Answer:** (d) 1
5. **Answer:** (b) $-\frac{\pi}{2}$
6. **Answer:** (d) $\frac{x}{\sqrt{1+x^2}}$
7. **Answer:** (c) 0
8. **Answer:** (c) $\frac{\pi}{4}$
9. **Answer:** (d) $-\frac{-\pi}{10}$
10. **Answer:** (a) $\frac{2\pi}{9}$

## Very Short Answer:

1. **Solution:**

$$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Hence, the principal value of sin−1 $\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$

2. **Solution:**

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

$$=\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{2\pi}{3}+\frac{\pi}{3}=\pi.$$

3. **Solution:**

$$\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\sec^{-1}(-2)=\frac{2\pi}{3}\in\left[0,\frac{\pi}{2}\right]-\left\{\frac{\pi}{2}\right\}$$

$$\therefore \tan^{-1}\left(\sqrt{3}\right)-\sec^{-1}(-2)$$

$$=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}.$$

4. **Solution:**

$$\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$$

$$=\tan^{-1}\left[2\cos\left(2\cdot\frac{\pi}{6}\right)\right]$$

$$\tan^{-1}(1)=\frac{\pi}{4}.$$

5. **Solution:**

$$\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$$

$$=\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)=-\frac{\pi}{2}$$

6. **Solution:**

$$\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow x=\frac{1}{3}$$

$$\left[\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right.$$

7. **Solution:**

$$\sec^{-1}(2)+\operatorname{cosec}^{-1}(y)=\frac{\pi}{2}$$

$$\Rightarrow y=2$$

$$\left[\because \sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}\right]$$

8. **Solution:**

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$\left[\because \sin^{-1}(-x)=-\sin^{-1}x\right]$$

$$=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\frac{\pi}{2}=1$$

9. **Solution:**

**L.H.S.** $=\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

$$=\cos\left(\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{3}{\sqrt{13}}\right)$$

$$=\cos\left[\cos^{-1}\left(\frac{4}{5}\cdot\frac{3}{\sqrt{13}}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{9}{13}}\right)\right]$$

$$=\frac{12}{5\sqrt{13}}-\frac{3}{5}\cdot\frac{2}{\sqrt{13}}=\frac{6}{5\sqrt{13}}=\textbf{R.H.S.}$$

**10. Solution:**

We have $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$

$$\Rightarrow\ \tan^{-1}\frac{x+y}{1-xy}=\frac{\pi}{4}\ \Rightarrow\ \frac{x+y}{1-xy}=\tan\frac{\pi}{4}$$

$$\Rightarrow\ \frac{x+y}{1-xy}=1\ \Rightarrow\ x+y=1-xy.$$

Hence $x + y + xy = 1$.

## Short Answer:

**1. Solution:**

$$\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)\text{ if }-\frac{\pi}{4}<x<\frac{\pi}{4}$$

$$\sin^{-1}\left(\sin x\cdot\frac{1}{\sqrt{2}}+\cos x\cdot\frac{1}{\sqrt{2}}\right)\text{ if }-\frac{\pi}{4}<x<\frac{\pi}{4}$$

$$=\sin^{-1}\left(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right)$$

$$\text{if }-\frac{\pi}{4}+\frac{\pi}{4}<x+\frac{\pi}{4}<\frac{\pi}{4}+\frac{\pi}{4}$$

$$=\sin^{-1}\left(\sin\left(x+\frac{\pi}{4}\right)\right)\text{ if }0<\left(x+\frac{\pi}{4}\right)<\frac{\pi}{2}$$

*i.e.* principal value

$$=x+\frac{\pi}{4}.$$

**2. Solution:**

Let **L.H.S.** = 0.

$$\text{Then }\sin\theta=\sin\left(\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right)$$

$$=\sin\left(\cos^{-1}\left(\frac{12}{13}\right)\right)\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$$

$$+\cos\left(\cos^{-1}\left(\frac{12}{13}\right)\right)\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$$

$$=\sqrt{1-\frac{144}{169}}\sqrt{1-\frac{9}{25}}+\frac{12}{13}\times\frac{3}{5}$$

$$=\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{3}{5}=\frac{20}{65}+\frac{36}{65}=\frac{56}{65}$$

$$\Rightarrow\ \theta=\sin^{-1}\frac{56}{65}.$$

Hence, $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}.$

**3. Solution:**

**L.H.S.** $=\sin^{-1}\frac{8}{17}+\cos^{-1}\frac{4}{5}$

$$=\tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4}$$

$$=\tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}=\tan^{-1}\frac{77}{36}=\cot^{-1}\left(\frac{36}{77}\right)$$

**= R.H.S.**

**4. Solution:**

$$\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x+1}{x}\right)=\tan^{-1}(-7)$$

$$\Rightarrow\ \tan^{-1}\left[\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\times\frac{x-1}{x}}\right]=\tan^{-1}(-7)$$

$$\Rightarrow\ \frac{x(x+1)+(x-1)^2}{x(x-1)-(x^2-1)}=-7$$

$\Rightarrow 2x^2 - 8x + 8 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x - 2)^2 = 0.$

Hence, $x = 2$.

**5. Solution:**

$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$

$$\Rightarrow\ \tan^{-1}\frac{2\sin x}{1-\sin^2 x}=\tan^{-1}(2\sec x)$$

$$\Rightarrow\ \tan^{-1}\left(\frac{2\sin x}{\cos^2 x}\right)=\tan^{-1}(2\sec x)$$

$\Rightarrow\ \tan^{-1}(2\sec x\tan x) = \tan^{-1}(2\sec x)$

$\Rightarrow\ 2\sec x\tan x = 2\sec x$

$\tan x = 1$ [$\because \sec x \neq 0$ ]

Hence, $x=\frac{\pi}{2}$

**6. Solution:**

We hace:

$$\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$\cos(\tan^{-1}x)=\sin\left(\sin^{-1}\frac{4}{5}\right)$$

$$\cos(\tan^{-1}x)=\frac{4}{5}$$

$$\Rightarrow \tan^{-1}x=\tan^{-1}\frac{3}{4}$$

Hence $x=\frac{3}{4}$

**7. Solution:**

Put $x = \cos\theta$ in R.H.S.

As $1/2 \le x \le 1$

R.H.S. = $\cos^{-1}(4\cos^3\theta - 3\cos\theta)$,

= $\cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x$ = L.H.S.

## Long Answer:

**1. Solution:**

$$\textbf{L.H.S.} =\cos^{-1}x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]$$

$$=\cos^{-1}x+\cos^{-1}\left[\frac{1}{2}\cdot x+\frac{\sqrt{3}}{2}\cdot\sqrt{1-x^2}\right]$$

$$=\theta+\cos^{-1}\left[\cos\frac{\pi}{3}\cdot\cos\theta+\sin\frac{\pi}{3}\cdot\sin\theta\right]$$

[Putting x = cos θ so that $\sqrt{1-x^2}$ =sin θ]

$$=\theta+\cos^{-1}\left[\cos\left(\frac{\pi}{3}-\theta\right)\right]$$

$$\theta+\left(\frac{\pi}{3}-\theta\right)=\frac{\pi}{3}= \textbf{R.H.S.}$$

**2. Solution:**

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right]$$

$$\left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$=\tan^{-1}\left[\frac{x(x+y)-y(x-y)}{y(x+y)+x(x-y)}\right]$$

$$=\tan^{-1}\left[\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right]$$

$$\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)=\tan^{-1}(1)=\frac{\pi}{4}.$$

**3. Solution:**

$$\textbf{L.H.S.} =\left(\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\frac{5+2}{10-1}+\tan^{-1}\left(\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\tan^{-1}\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}$$

$$=\tan^{-1}\frac{56+9}{72-7}=\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

**= R.H.S.**

**4. Solution:**

$$2\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(2\cdot\frac{3}{5}\sqrt{1-\frac{9}{25}}\right)$$

$$\left[\because 2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)\right]$$

$$=\sin^{-1}\left(2\times\frac{3}{5}\times\frac{4}{5}\right)=\sin^{-1}\left(\frac{24}{25}\right)$$

$$=\tan^{-1}\left(\frac{24}{7}\right) \qquad ....(i)$$

Now, **L.H.S.** $=2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$$=\tan^{-1}\left(\frac{24}{7}\right)-\tan^{-1}\left(\frac{17}{31}\right) \qquad \text{[Using (i)]}$$

$$=\tan^{-1}\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}=\tan^{-1}\frac{744-119}{217+408}$$

$$=\tan^{-1}\frac{625}{625}=\tan^{-1}(1)=\frac{\pi}{4}\ \textbf{=R.H.S.}$$

## Assertion and Reason Answers:

**1.** (d)

**Solution :**

Given R = {(1,1), (1,3), (3,1), (3,3), (3,5)}

We know that Relation 'R' is reflexive on set A if ∀a∈A,(a,a)∈R

Here set A={1,3,5}

(1,1)∈R,(3,3)∈R

but (5,5)∉R

∴R is not reflexive

∴ Assertion A is false By definition of transition Relation, It is clear that given Reason R is true. Hence option (d) is the correct answer.

**2.** (c)

**Solution:**

Given R={(a, b):|a−b|<2}

= {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1), (2, 3), (3, 2), (3, 4). (4, 3), (4, 5), (5, 4)}.

Here (1,1) ∈ R, (2, 2) ∈ R, (3, 3) ∈ R, (4, 4) ∈ R, (5, 5) ∈ R

⇒ Relation R is reflexive on set A = {1,2,3,4,5}

∴ Assertion A is true We know that relation R is reflexive if (a, a) ∈R ∀ a ∈A

∴ Given Reason R is false Hence option (c) is the correct answer.

# Matrices 3

## Top Terms

1. A matrix is an ordered rectangular array of numbers (real or complex) or functions or names or any type of data. The numbers or functions are called the elements or the entries of the matrix.
2. The horizontal lines of elements constitute the rows of the matrix and the vertical lines of elements constitute the columns of the matrix.
3. Each number or entity in a matrix is called its element.
4. If a matrix contains m rows and n columns, then it is said to be a matrix of the order m × n (read as m by n).
5. The total number of elements in a matrix is equal to the product of its number of rows and number of columns.
6. A matrix is said to be a column matrix if it has only one column.
7. $A = [a_{ij}]_{m \times 1}$ matrix is said to be a row matrix if it has only one row.
8. A matrix is said to be a row matrix if it has only one row.
9. $B = [b_{ij}]_{1 \times n}$ is row matrix of order 1 × n.
10. **Rectangular matrix:** A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix.
11. A matrix each of whose elements is zero is called a zero matrix or null matrix.
12. A matrix in which the number of rows is equal to the number of columns is said to be a square matrix. A matrix of order 'm × n' is said to be a square matrix if m = n and is known as a square matrix of order 'n'.
13. A square matrix which has every non—diagonal element as zero is called a diagonal matrix.
14. A square matrix $A = [a_{ij}]_{m \times m}$ is said to be a diagonal matrix if all its non-diagonal elements are zero, i.e., a matrix $A = [a_{ij}]_{m \times m}$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$.
15. A square matrix in which the elements in the diagonal are all 1 and the rest are all zero is called an identity matrix. A square matrix $A = [a_{ij}]_{n \times n}$ is an matrix if

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

16. A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is a square matrix $B = [b_{ij}]_{n \times n}$ is said to be a scaler matrix if $b_{ij} = 0$ when $i \neq j$ and $b_{ij} = k$ when $i = j$ for some constant k.
17. **Upper triangular matrix:** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$. In an upper triangular matrix, all elements below the main diagonal are zero.
18. **Lower triangular matrix:** A square matrix A = [a„] is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. In a lower triangular matrix, all elements above the main diagonal are zero.
19. Two matrices are said to be equal if they are of the same order and have the same corresponding elements.
20. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of the same order. Each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

**21.** If A is a matrix, then its transpose is obtained by interchanging its rows and columns. Transpose of a matrix A is denoted by $A^t$. If A = $[a_{ij}]$ be an n × m matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or ($A^T$). Theft is, $(A^T)_{ij} = a_{ji}$ for all l = 1, 2, ... m; j = 1, 2, ... n.

**22.** If A = $[a_{ij}]_{nxn}$ is an n × n matrix such that $A^T$ = A, then A is called a symmetric matrix. In a symmetric matrix, $a_{ij} = a_{ji}$ for all i and j.

**23.** If A = $[a_{ij}]_{nxn}$ is an n × n matrix such that $A^T$ = -A, then A is called a skew-symmetric matrix. In a skew-symmetric matrix, $a_{ij} = a_{ji}$.

**24.** All main diagonal elements of a skew-symmetric matrix are zero.

**25.** Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

**26.** All positive integral powers of a symmetric matrix are symmetric.

**27.** All odd positive integral powers of a skew-symmetric matrix are skew-symmetric.

**28.** Let A and B be two square matrices of the order n such that AB = BA = I.

Then A is called the inverse of B and is denoted by B = $A^{-1}$. If B is the inverse of A, then A is also the inverse of B.

**29.** If A and B are two invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$.

## Top Concepts

**1.** The order of a matrix gives the number of rows and columns present in the matrix.

**2.** If a matrix A has m rows and n columns, then it is denoted by A = $[a_{ij}]$m × n. Here $a_{ij}$ is i-j th or (i, j)$^{th}$ element of the matrix.

**3.** The simplest classification of matrices is based on the order of the matrix.

**4.** In case of a square matrix, the collection of elements $a_{11}$. $a_{22}$ and so on constitute the Principal Diagonal or simply the diagonal of the matrix.

**5.** The diagonal is defined only in the case of square matrices.

$$\begin{bmatrix} a_{11} & a_{12} & .... & .... & a_{1n} \\ a_{21} & a_{22} & .... & .... & a_{2n} \\ .... & .... & .... & .... & .... \\ a_{i1} & a_{i2} & .... & .... & a_{in} \\ .... & .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & .... & a_{nn} \end{bmatrix}_{nxn}$$

**6.** Two matrices of the same order are comparable matrices.

**7.** If A = $[a_{ij}]_{m \times n}$ and B = = $[b_{ij}]_{m \times n}$ are two matrices of the order m × n, then their sum is defined as a matrix C = $[c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$.

**8.** Two matrices can be added (or subtracted) if they are of the same order.

**9.** For multiplying two matrices A and B, the number of columns in A must be equal to the number of rows in B.

**10.** If A = $[a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

Hence, kA = $[ka_{ij}]_{m \times n}$

**11.** If A = $[a_{ij}]_{m \times n}$ and B = = $[b_{ij}]_{m \times n}$ are are two matrices, then their difference is represented as A – B = A + (-1)B.

**12. Properties of matrix addition**

- Matrix addition is commutative, i.e., A + B = B + A
- Matrix addition is associative, i.e. (A + B) + C = A + (B + C)

- Existence of additive identity: NuIl matrix is the identity with respect to addition of matrices.

  Given a matrix $A = [a_{ij}]_{m \times n}$, there will be a corresponding null matrix O of the same order such that A + O = O + A = A
- The existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$ be any matrix, then there exists another matrix $-A = -[a_{ij}]_{m \times n}$ Such that

$$A + (-A) = (-A) + A = O.$$

**13. Cancellation law:** If A, B and C are three matrices of the same order, then

$$A + B = A + C \Rightarrow B = C$$

and

$$B + A = C + A \Rightarrow B = C$$

**14.** Properties of scalar multiplication of matrices

If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices, and k and L are real number, then

i. k(A + B) = kA + kB

ii. (k + l)A = kA + IA

iii. $k(A + B) = k([a_{ij}]+[b_{ij}]) = k[a_{ij}] + k[b_{ij}] = kA + kB$

iv. $(k + L)A = (k + L)\ [a_{ij}] = [(k + L)a_{ij}] = k[a_{ij}] + L[a_{ij}] = kA + LA$

**15.** If $A = [a_{ij}]_{mxp}$, $B = [b_{ij}]_{pxn}$ are two matrices, then their product AB is given by $C = [c_{ij}]_{mxn}$ such that

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + .... + a_{ip}b_{pj}$$

In order to multiply two matrices A and B, the number of columns in A = number of rows in B.

**16.** Properties of Matrix Multiplication

Commutative law does not hold in matrices, whereas associative and distributive laws hold for matrix multiplication.

i. In general, AB ≠ BA

ii. Matrix multiplication is associative A(BC) = (AB)C

iii. Distributive laws:

$$A(B + C) = AB + AC$$
$$(A + B)C = AC + BC$$

**17.** The multiplication of two non-zero matrices can result in a null matrix.

**18.** If A is a square matrix, then we define A1 = A and $A^{n+1} = A^n . A$

**19.** If A is a square matrix, $a_0, a_1, a_2, ..., a_n$ are constants, then $a_0A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_{n-1}A + a_n$ is called a matrix polynomial.

**20.** If A, B and C are matrices, then AB = AC, A ≠ 0 ⇒ B = C.

In general, the cancellation law is not applicable in matrix multiplication.

**21.** Properties of transpose of matrices

i. If A is a matrix, then (AT)T = A

ii. $(A + B)^T = A^T + B^T$

iii. $(kB)^T = kB^T$, where k is any constant.

**22.** If A and B are two matrices such that AB exists, then $(AB)^T = B^T A^T$.

**23.** If A, B and C are two matrices such that AB exists, then $(ABC)^T = C^T B^T A^T$.

**24.** Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix, i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ for any square matrix A.

25. A square matrix A is called an orthogonal matrix when $AA^T = A^TA = I$.

26. A null matrix is both symmetric and skew symmetric.

27. Multiplication of diagonal matrices of the same order will be commutative.

28. There are six elementary operations on matrices—three on rows and three on columns. The first operation is interchanging the two rows, i.e., $R_i \leftrightarrow R_j$ implies that the $i^{th}$ row is interchanged with the $j^{th}$ row. The two rows are interchanged with one another and the rest of the matrix remains the same.

29. The second operation on matrices is to multiply a row with a scalar or a real number, i.e., $R_i \leftrightarrow kR_i$ that $i^{th}$ row of a matrix A is multiplied by k.

30. The third operation is the addition to the elements of any row, the corresponding elements of any other row multiplied by any non-zero number, i.e., $R_i \to R_i + kR_i$ k multiples of the $j^{th}$ row elements are added to the $i^{th}$ row elements.

31. Column operation on matrices ar,
    i. Interchanging the two columns: $C_r \leftrightarrow C_k$ indicates that the $r^{th}$ column is interchanged with the $k^{th}$ column.
    ii. Multiply a column with a non-zero constant, i.e., $C_i \to kC_i$
    iii. Addition of a scalar multiple of any column to another column, i.e. $C_i \to C_j + kC_j$

32. Elementary operations help in transforming a square matrix to an identity matrix.

33. The inverse of a square matrix, if it exists, is unique.

34. The inverse of a matrix can be obtained by applying elementary row operations on the matrix A = IA. In order to use column operations, write A = AI.

35. Either of the two operations—row or column—can be applied. Both cannot be applied simultaneously.

36. For any square matrix A with real number entries, A + A' is a symmetric matrix and A — A' is a skew-symmetric matrix.

    Laws of algebra are not applicable to matrices, i.e.

$$(A + B)^2 \neq A^2 + 2AB + B^2$$

and

$$(A + B)(A - B) \neq A^2 - B^2$$

## Top Formulae

1. An $m \times n$ matrix is a square matrix if $m = n$.

2. $A = [a_{ij}] = [b_{ij}] = B$ if

   (i) A and B are of the same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.

3. $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$

4. $-A = (-1)A$

5. $A - B = A + (-1)B$

6. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, then $AB = C = [c_{ik}]_{m \times p}$,

   where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{ij}$

7. Elementary operations of a matrix are as follows:
   i. $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
   ii. $R_i \to kR_i$ or $C_i \to kC_i$
   iii. $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

**Class : 12th Maths**
**Chapter- 3 : Matrices**

## Matrices

### Definition and its types

A matrix of order m×n is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A=[a_{ij}]_{m\times n}$ is given by $A=\begin{bmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ a_{m1} & a_{m2} & ...a_{mn}\end{bmatrix}_{m\times n}$

- Column matrix : It is of the form $[a_{ij}]_{m\times 1}$
- Row matrix : It is of the form $[a_{ij}]_{1\times n}$
- Square matrix : Here, $m = n$ (no. of rows = no. of columns)
- Diagonal matrix : All non-diagonal entries are zero i.e $a_{ij} = 0 \forall i \neq j$
- Scalar matrix : $a_{ij} = 0, i \neq j$ and $a_{ij} = k$ (Scalar), $i = j$
- Identity matrix : $a_{ij} = 0, i \neq j$ and $a_{ij} = 1, i = j$
- Zero matrix : All entries are zero.

### Types of Matrix

If $A=[a_{ij}]_{m\times n}$, then its transpose $A'$ $(A^T)=[a_{ji}]_{n\times m}$ i.e. if $A=(2\ \ 1)$ then $A^T=\begin{pmatrix}1\\2\end{pmatrix}$.

Also, $(A')'=A$, $(kA)'=kA'$, $(A+B)'=A'+B'$, $(AB)'=B'A'$.

- A is symmetric matrix if $A=A'$ i.e. $A'=-A$.
- A is skew – symmetric if $A=-A'$ i.e. $A'=-A$.
- A is any matrix, then–

$A=\frac{1}{2}\{(A+A')+(A-A')\}$ = sum of a symmetric and a skew-symmetric matrix.
(A + A') : S.M; (A − A') : Skew, S.M.

For eg if $A=\begin{pmatrix}2 & 8\\ 6 & 4\end{pmatrix}$, then $A=\frac{1}{2}\left\{\begin{pmatrix}2 & 7\\ 7 & 4\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}$.

### Equality of Two Matrix

$A=[a_{ij}]=[b_{ij}]=B$ if, A and B are of same order and $a_{ij}=b_{ij}\ \forall\ i$ and $j$.

### Elementary operations on a matrix

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
$R_i \to kR_i$ or $C_i \to kC_i$
$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

If A, B are square matrices such that AB = BA = I then $B=A^{-1}$ i.e., A is the inverse of B and vice-versa.
Inverse of a square matrix, if it exists, is unique
For eg: If $A=\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}$, then after $R_1 \leftrightarrow R_2$, A becomes $\begin{pmatrix}3 & 4\\ 1 & 2\\ 5 & 6\end{pmatrix}$
If A and B are invertible matrices of the same order, then $(AB)^{-1}=B^{-1}A^{-1}$. By elementary transformations, we can convert A=IA to I $A^{-1}A$. This is one process of finding the inverse of a given square matrix A.

### Operations on matrices

#### Addition

If A, B are two matrices of same order, then $A+B=[a_{ij}+b_{ij}]$ The addition of A and B follows: $A+B=B+A$, $(A+B)+C=A+(B+C)$, $-A=(-1)A$, $k(A+B)=kA+kB$, k is scalar and $(k+l)A=kA+lA$, k and l are constants.

#### Multiplication

- If $A=\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix}$, $B=\begin{pmatrix}-3 & 2\\ -4 & 5\end{pmatrix}$ then $A+B=\begin{pmatrix}-1 & 5\\ -2 & 9\end{pmatrix}$
- If $A=(2\ 3)_{1\times 2}$, $B=\begin{pmatrix}4\\5\end{pmatrix}_{2\times 1}$, then $AB=(2\times 4+3\times 5)=(2\ 3)_{1\times 1}$

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then $AB=C=[C_{ik}]_{m\times p}$, $[C_{jk}]=\sum_{j=1}^{n} a_{ij}b_{jk}$ . Also, $A(BC)=(AB)C$, $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$, but $AB \neq BA$ (always).

# Important Questions

## Multiple Choice questions-

1. If A = $[a_{ij}]_{m \times n}$ is a square matrix, if:
   (a) m < n
   (b) m > n
   (c) m = n
   (d) None of these.
2. Which of the given values of x and y make the following pair of matrices equal:
   $$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}\begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$
   (a) x = $-\frac{1}{3}$, y = 7
   (b) Not possible to find
   (c) y = 7, x = $-\frac{2}{3}$
   (d) x = $-\frac{1}{3}$, y = $-\frac{2}{3}$
3. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is
   (a) 27
   (b) 18
   (c) 81
   (d) 512

   Assume X, Y, Z, W and P are matrices of order 2 × n, 3 × 1, 2 × p, n × 3 and p × k respectively. Now answer the following (4-5):
4. The restrictions on n, k and p so that PY + WY will be defined are
   (a) k = 3, p = n
   (b) k is arbitrary, p = 2
   (c) p is arbitrary
   (d) k = 2, p = 3.
5. If n =p, then the order of the matrix 7X - 5Z is:
   (a) p × 2
   (b) 2 × n
   (c) n × 3
   (d) p × n.
6. If A, B are symmetric matrices of same order, then AB – BA is a
   (a) Skew-symmetric matrix
   (b) Symmetric matrix
   (c) Zero matrix
   (d) Identity matrix.
7. If $A=\begin{bmatrix} \cos\alpha - \sin\alpha \\ \sin\alpha \cos\alpha \end{bmatrix}$ then A + A′ = I, the value of a is:
   (a) $\frac{\pi}{6}$
   (b) $\frac{\pi}{3}$
   (c) π
   (d) $\frac{3\pi}{2}$
8. Matrices A and B will be inverse of each other only if:
   (a) AB = BA
   (b) AB – BA = 0
   (c) AB = 0, BA = I
   (d) AB = BA = I.
9. If $A=\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2$ = I, then
   (a) $1 + \alpha^2 + \beta\gamma = 0$
   (b) $1 - \alpha^2 + \beta\gamma = 0$
   (c) $1 - \alpha^2 - \beta\gamma = 0$
   (d) $1 + \alpha^2 - \beta\gamma = 0$
10. If a matrix is both symmetric and skew-symmetric matrix, then:
   (a) A is a diagonal matrix
   (b) A is a zero matrix
   (c) A is a square matrix
   (d) None of these.

## Very Short Questions:

1. If a matrix has 8 elements, what are the possible orders it can have.
2. Identity matrix of orders n is denoted by.
3. Define square matrix
4. The no. of all possible metrics of order 3 × 3 with each entry 0 or 1 is
5. Write (1) a33, a12 (ii) what is its order
   $$A=\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$
6. Two matrices A = $a_{ij}$ and B = $b_{ij}$ are said to be equal if
7. Define Diagonal matrix.

8. Every diagonal element of a skew symmetric matrix is

9. If $A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$ then A + A′ = I, Find α

10. $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$. Find A + A′.

## Short Questions:

1. Write the element $a_{23}$ of a 3 x 3 matrix A = [$a_{ij}$] whose elements $a_{ij}$ are given by: $\frac{|i-j|}{2}$

2. For what value of $x$ is

$$\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=0?$$

3. Find a matrix A such that 2A – 3B + 5C = 0,

Where $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

If $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & -\cos\alpha\end{pmatrix}$, then for what value of 'a' A an identity matrix?

4. Find the values of x, y, z and t, if:

$$2\begin{bmatrix}x & z\\ y & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\ 4 & 6\end{bmatrix}$$

5. If $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$, then find $(A^2-5A)$.

6. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find $k$ so that $A^2 = 5A + bkl$.

7. If A and B are symmetric matrices, such that AB and BA are both defined, then prove that AB – BA is a skew symmetric matrix.

## Long Questions:

1. Find the values of a, b, c and d from the following equation:

$$\begin{bmatrix}2a+b & a-2b\\ 5c-d & 4c+3d\end{bmatrix}=\begin{bmatrix}4 & -3\\ 11 & 24\end{bmatrix}$$

2. If $\begin{bmatrix}9 & -1 & 4\\ -2 & 1 & 3\end{bmatrix}=A+\begin{bmatrix}1 & 2 & -1\\ 0 & 4 & 9\end{bmatrix}$, then find the matrix A.

3. If $A=\begin{bmatrix}2 & 2\\ -3 & 1\\ 4 & 0\end{bmatrix}$ $B=\begin{bmatrix}6 & 2\\ 1 & 3\\ 0 & 4\end{bmatrix}$, find the matrix C such that A + B + C is a zero matrix.

4. If $A=\begin{bmatrix}8 & 0\\ 4 & -2\\ 3 & 6\end{bmatrix}$ $B=\begin{bmatrix}2 & -2\\ 4 & 2\\ -5 & 1\end{bmatrix}$, then find the matrix '$X$', of order 3 × 2, such that $2A + 3x = 5B$.

## Assertion and Reason Questions:

**1.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

**Assertion(A):** $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is an identity matrix.

**Reason (R):** A matrix A=[$a_{ij}$] is an identity matrix if $a_{ij}=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if} \neq j\end{cases}$.

**2.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

**Assertion (A):** Matrix $\begin{bmatrix}1\\ 5\\ 2\end{bmatrix}$ is a column matrix.

**Reason(R):** A matrix of order m × 1 is called a column matrix.

## Case Study Questions:

**1.** Three shopkeepers A, B and C go to a store to buy stationary. A purchase 12 dozen notebooks, 5 dozen pens and 6 dozen pencils. B purchases 10 dozen notebooks, 6 dozen pens and 7 dozen pencils. C purchases 11 dozen notebooks, 13 dozen pens and 8 dozen pencils. A notebook costs ₹ 40, a pen costs ₹ 12 and a pencil costs ₹ 3.

Based on the above information, answer the following questions.

(i) The number of items purchased by shopkeepers A, B and C represented in matrix form as:

(a) $\begin{array}{c} \textit{Notebooks} \quad \textit{Pens} \quad \textit{Pencils} \\ \begin{bmatrix} 144 & 60 & 72 \\ 120 & 720 & 84 \\ 132 & 156 & 96 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$

(b) $\begin{array}{c} \textit{Notebooks} \quad \textit{Pens} \quad \textit{Pencils} \\ \begin{bmatrix} 144 & 72 & 60 \\ 120 & 84 & 72 \\ 132 & 156 & 96 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$

(c) $\begin{array}{c} \textit{Notebooks} \quad \textit{Pens} \quad \textit{Pencils} \\ \begin{bmatrix} 144 & 72 & 72 \\ 120 & 156 & 84 \\ 132 & 84 & 96 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$

(d) $\begin{array}{c} \textit{Notebooks} \quad \textit{Pens} \quad \textit{Pencils} \\ \begin{bmatrix} 144 & 60 & 60 \\ 120 & 84 & 72 \\ 132 & 156 & 96 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$

(ii) If Y represents the matrix formed by the cost of each item, then XY equals.

(a) $\begin{bmatrix} 5741 \\ 6780 \\ 8040 \end{bmatrix}$

(b) $\begin{bmatrix} 6696 \\ 5916 \\ 7440 \end{bmatrix}$

(c) $\begin{bmatrix} 5916 \\ 6696 \\ 7440 \end{bmatrix}$

(d) $\begin{bmatrix} 6740 \\ 5740 \\ 8140 \end{bmatrix}$

(iii) Bill of A is equal to:
a. ₹ 6740
b. ₹ 8140
c. ₹ 5740
d. ₹ 6696

(iv) If $A^2 = A$, then $(A + 1)^3 - 7A =$
a. A
b. A - I
c. I
d. A + I

(v) If A and B are 3 × 3 matrices such that $A^2 - B^2 = (A - B)(A + B)$, then
a. Either A or B is zero matrix.
b. Either A or B is unit matrix.
c. A = B
d. AB = BA

**2.** Consider 2 families A and B. Suppose there are 4 men,4 women and 4 children in family A and 2 men, 2 women and 2 children in family B. The recommend daily amount of calories is 2400 for a man, 1900 for a woman, 1800 for a children and 45 grams of proteins for a man, 55 grams for a woman and 33 grams for children.

Based on the above information, answer the following questions.

(i) The requirement of calories and proteins for each person in matrix form can be represented as:

(a) $\begin{array}{c} \quad\quad\quad \textit{Calorise} \quad \textit{Proteins} \\ \begin{matrix} \textit{Man} \\ \textit{Woman} \\ \textit{Children} \end{matrix} \begin{bmatrix} 2400 & 45 \\ 1900 & 55 \\ 1800 & 33 \end{bmatrix} \end{array}$

(b) $\begin{array}{c} \quad\quad\quad \textit{Calorise} \quad \textit{Proteins} \\ \begin{matrix} \textit{Man} \\ \textit{Woman} \\ \textit{Children} \end{matrix} \begin{bmatrix} 1900 & 55 \\ 2400 & 45 \\ 1800 & 33 \end{bmatrix} \end{array}$

(c) $\begin{array}{c} \quad\quad\quad \textit{Calorise} \quad \textit{Proteins} \\ \begin{matrix} \textit{Man} \\ \textit{Woman} \\ \textit{Children} \end{matrix} \begin{bmatrix} 1800 & 33 \\ 1900 & 55 \\ 2400 & 45 \end{bmatrix} \end{array}$

(d) $\begin{array}{c} \quad\quad\quad \textit{Calorise} \quad \textit{Proteins} \\ \begin{matrix} \textit{Man} \\ \textit{Woman} \\ \textit{Children} \end{matrix} \begin{bmatrix} 2400 & 33 \\ 1900 & 55 \\ 1800 & 45 \end{bmatrix} \end{array}$

(ii) Requirement of calories of family A is:

a. 24000

b. 24400

c. 15000

d. 15800

(iii) Requirement of proteins for family B is:

a. 560 grams

b. 332 grams

c. 266 grams

d. 300 grams

(iv) If A and Bare two matrices such that AB = B and BA = A, then $A^2 + B^2$ equals.

a. 2AB

b. 2BA

c. A + B

d. AB

(v) If $A=(a_{ij})_{m\times n}, B=(b_{ij})_{n\times p}$ and $C=(c_{ij})_{p\times q}$ then the product (BC) A is possible only when.

a. m = q

b. n = q

c. p = q

d. m = p

# Answer Key

## Multiple Choice questions-

1. **Answer:** (c) m = n
2. **Answer:** (b) Not possible to find
3. **Answer:** (d) 512.
4. **Answer:** (a) k = 3, p = n
5. **Answer:** (b) 2 × n
6. **Answer:** (a) Skew-symmetric matrix
7. **Answer:** (a) $\frac{\pi}{6}$
8. **Answer:** (d) AB = BA = I.
9. **Answer:** (c) $1 - \alpha^2 - \beta\gamma = 0$
10. **Answer:** (b) A is a zero matrix

## Very Short Answer:

**1. Solution:**

$1\times 8,\ \ 8\times 1,\ \ 4\times 2,\ \ 4\times 4$

**2. Solution:** $I_n$

**3. Solution:** A matrix in which the no. of rows are equal to no. of columns i.e. m = n

**4. Solution:** $512=2^9$

**5. Solution:**

(i) $a_{33} = 9, a_{12} = 4$

(ii) 3 × 3

**6. Solution:** They are of the same order.

**7. Solution:** A square matrix in which every non – diagonal element is zero is called diagonal matrix.

**8. Solution:** Zero.

**9. Solution:**

$$A + A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix}$$

$A + A' = I$ (Given)

$$\begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2\cos\alpha = 1$$

$$\cos\alpha = \frac{1}{2}$$

$$\cos\alpha = \cos\frac{\pi}{3}$$

$$\alpha = \frac{\pi}{3}$$

**10. Solution:**

$$A = A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$$

## Short Answer:

**1. Solution:**

We have $\left[a_{ij}\right]=\frac{|i-j|}{2}$

$\therefore\ a_{23}=\frac{|2-3|}{2}=\left|\frac{-1}{2}\right|=\frac{1}{2}$

**2. Solution:**

We have

$$\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\2 & 0 & 1\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\2\\x\end{bmatrix}=0$$

$$\begin{bmatrix}1+4+12+0+00+2+2\end{bmatrix}\begin{bmatrix}0\\2\\x\end{bmatrix}=0$$

$$\begin{bmatrix}6 & 2 & 4\end{bmatrix}\begin{bmatrix}0\\2\\x\end{bmatrix}=0$$

$\Rightarrow [0+4+4x]=0$

$\Rightarrow [4+4x]=[0]$

$\Rightarrow\ 4+4x=0$

Hence, $x=-1$

**3. Solution:**

Here, 2A - 3B + 5C = 0

⇒ 2A = 3B - 5C

$$\Rightarrow\ 2A=3\begin{bmatrix}-2 & 2 & 0\\3 & 1 & 4\end{bmatrix}-5\begin{bmatrix}2 & 0 & -2\\7 & 1 & 6\end{bmatrix}$$

$$=\begin{bmatrix}-6 & 6 & 0\\9 & 3 & 12\end{bmatrix}+\begin{bmatrix}-10 & 0 & 10\\-35 & -5 & -30\end{bmatrix}$$

$$=\begin{bmatrix}-6-10 & 6+0 & 0+10\\9-35 & 3-5 & 12-30\end{bmatrix}$$

$$=\begin{bmatrix}-16 & 6 & 10\\-26 & -2 & -18\end{bmatrix}$$

$$\text{Hence, } A=\begin{bmatrix}-8 & 3 & 5\\-13 & -1 & -9\end{bmatrix}$$

**4. Solution:**

$$\text{Here } A=\begin{pmatrix}\cos\alpha-\sin\alpha\\\sin\alpha-\cos\alpha\end{pmatrix}$$

Now $A=I=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ when cos α = 1 and sin α = 0.

Hence α = 0.

**5. Solution:**

We have:

$$2\begin{bmatrix}x & z\\y & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\4 & 6\end{bmatrix}$$

$$\Rightarrow\ \begin{bmatrix}2x & 2z\\2y & 2t\end{bmatrix}+\begin{bmatrix}3 & -3\\0 & 6\end{bmatrix}=\begin{bmatrix}9 & 15\\12 & 18\end{bmatrix}$$

$$\Rightarrow\ \begin{bmatrix}2x+3 & 2z-3\\2y & 2t+6\end{bmatrix}=\begin{bmatrix}9 & 15\\12 & 18\end{bmatrix}$$

⇒ 2x + 3 = 9 ............ (1)

2z – 3 = 15 ............ (2)

2y = 12 ............ (3)

2t + 6 = 18 ............ (4)

From (1), ⇒ 2x = 9 – 3

⇒ 2x = 6

⇒ x = 3.

From (3) 2y = 12

⇒ y = 6.

From (2), ⇒ 2z – 3 = 15

⇒ 2z = 18

⇒ z = 9.

From (4), 2t + 6 = 18

⇒ 2t = 12

⇒ t = 6.

Hence, x = 3, y = 6, z = 9 and t = 6.

**6. Solution:**

$$\text{We have } A=\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$$

Then $A^2$ = AA

$$=\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}4+0+1 & 0+0-1 & 2+0+0\\4+2+3 & 0+1-3 & 2+3+0\\2-2+0 & 0-1-0 & 1-3+0\end{bmatrix}$$

$$=\begin{bmatrix}5 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix}$$

$$\therefore A^2 - 5A = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1+5 & -2-0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}.$$

**7. Solution:**

We have: $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} \quad ....(1)$$

$$\text{Also, } 5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} \quad ....(2)$$

$$\text{and } kI = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \quad ....(3)$$

$$\therefore A^2 = 5A + kI$$

$$\Rightarrow \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

[Using (1), (2) & (3)]

$$\Rightarrow \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 15+k & 5 \\ -5 & 10+k \end{bmatrix}$$

⇒ 8 = 15 + k and 3 = 10 + k

⇒ k = -1 and k = -7.

Hence, k – (-7).

**8. Solution:**

Since A and B are symmetric matrices,

∴ A' = A and B' = B ...(1)

Now, (AB – BA)'= (AB)' – (BA)'

= B'A' – A'B'

= BA – AB [Using (1)]

= – (AB – BA).

Hence, AB – BA is a skew-symmetric matrix.

## Long Answer:

**1. Solution:**

We have:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Comparing the corresponding elements of two given matrices, we get:

2a + b = 4 ...(1)

a-2b = – 3 ...(2)

5c-d = 11 ...(3)

4c + 3d = 24 ...(4)

Solving (1) and (2):

From (1),

b = 4 – 2a ...(5)

Putting in (2), a – 2 (4 – 2a) = – 3

⇒ a – 8 + 4a = -3

⇒ 5a = 5

⇒ a = 1.

Putting in (5),

b = 4 – 2(1) = 4 – 2 = 2.

Solving (3) and (4):

From (3),

d = 5c- 11 ...(6)

Putting in (4),

4c+ 3 (5c- 11) = 24

⇒ 4c + 15c – 33 = 24

⇒ 19c = 57

⇒ c = 3.

Putting in (6),

d = 5 (3) – 11 = 15 – 11 = 4.

Hence, a = 1, b = 2, c = 3 and d = 4.

**2. Solution:**

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Then $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+1 & a_{12}+2 & a_{13}-1 \\ a_{21}+0 & a_{22}+4 & a_{23}+9 \end{bmatrix}$$

Comparing:

9 = $a_{11}$ + 1 – 1 = $a_{12}$ + 2,

4 = $1_{13}$ – 1, -2 = $a_{21}$

1 = $a_{22}$ + 4, and 3 = $a_{23}$ + 9

$a_{11} = 8, a_{12} = -3,$

$a_{13} = 5, a_{21} = -2$

$a_{22} = -3$, and $a_{23} = -6$.

Hence, $A = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

**3. Solution:**

Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$

Then A + B + C = 0

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ -3 & 1 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 3 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+6 & 2+2 \\ -3+1 & 1+3 \\ 4+0 & 0+4 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+6+c_{11} & 2+2+c_{12} \\ -3+1+c_{21} & 1+3+c_{22} \\ 4+0+c_{31} & 0+4+c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8+c_{11} & 4+c_{12} \\ -2+c_{21} & 4+c_{22} \\ 4+c_{31} & 4+c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing:

$8 + c_{11} = 0 \Rightarrow c_{11} = -8,$

$4 + C_{12} = 0 \Rightarrow C_{12} = -4,$

$-2 + C_{21} = 0 \Rightarrow C_{21} = 2$

$4 + C_{22} = 0 \Rightarrow C_{22} = -4,$

$4 + c_{31} = 0 \Rightarrow C_{31} = -4$

and $4 + c_{32} = 0 \Rightarrow C_{32} = -4$.

Hence, $C = \begin{bmatrix} -8 & -4 \\ 2 & -4 \\ -4 & -4 \end{bmatrix}$

**4. Solution:**

We have: 2A + 3X = 5B

⇒ 2A + 3X-2A = 5B-2A

⇒ 2A-2A + 3X = 5B-2A

⇒ (2A – 2A) + 3X = 5B – 2A

⇒ O + 3X = 5B – 2A

[ ∵ – 2A is the inverse of2A]

⇒ 3X = 5B – 2A.

[ ∵ O is the additive identity]

Hence, $X = \frac{1}{3}(5B - 2A)$

$$= \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & -10/3 \\ 4 & 14/3 \\ -31/3 & -7/3 \end{bmatrix}$$

## Assertion and Reason Answers:

**1.** (d) A is false and R is true.

**Solution:**

We know that, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an indentity matrix

∴ Given Assertion [A] is false We know that for identity matrix $a_{ij} = 1$, $if i = j$ and $a_{ij} = 0$, $if i \neq j$

∴ Given Reason (R) is true Hence option (d) is the correct answer.

**2.** a) Both A and R are true and R is the correct explanation of A.

**Solution:**

We know that order of column matrix is always $m \times 1$

$\therefore \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

is column matrix.

⇒ Assertion (A) is true Also Reason (R) is true and is correct explanation of A. Hence option (a) is the correct answer.

## Case Study Answers:

**1. Answer:**

i. (a)

| | *Notebooks* | *Pens* | *Pencils* |
|---|---|---|---|
| *A* | 144 | 60 | 72 |
| *B* | 120 | 720 | 84 |
| *C* | 132 | 156 | 96 |

**Solution:**

$$X=\begin{array}{c} \\ \end{array}\begin{array}{c}\textit{Notebooks}\quad \textit{Pens}\quad \textit{Pencils}\end{array}$$

$$X=\begin{bmatrix}144 & 60 & 72\\ 120 & 720 & 84\\ 132 & 156 & 96\end{bmatrix}\begin{matrix}A\\ B\\ C\end{matrix}$$

ii. (b) $\begin{bmatrix}6696\\ 5916\\ 7440\end{bmatrix}$

**Solution:**

Since, $Y=\begin{bmatrix}40\\ 12\\ 3\end{bmatrix}\begin{matrix}\textit{Notebook}\\ \textit{Pens}\\ \textit{Pencil}\end{matrix}$

$$\therefore XY=\begin{bmatrix}144 & 60 & 72\\ 120 & 72 & 84\\ 132 & 156 & 96\end{bmatrix}\begin{bmatrix}40\\ 12\\ 3\end{bmatrix}$$

$$=\begin{bmatrix}5760+720+216\\ 4800+846+252\\ 5280+1872+288\end{bmatrix}=\begin{bmatrix}6696\\ 5916\\ 7440\end{bmatrix}$$

iii. (d) ₹ 6696

**Solution:**

Bill of A is ₹ 6696.

iv. (c) I

**Solution:**

$(A + I)^2 = A^2 + 2A + I = 3A + I$

$\Rightarrow (A + 1)^3 = (3A + I)(A + I)$

$= 3A^2 + 4A + I = 7A + I$

$\therefore (A + I)^3 - 7A = I$

v. (d) AB = BA

**Solution:**

$A^2 - B^2 = (A - B)(A + B) = A^2 + AB - BA - B^2$

$\therefore AB = BA$

**2. Answer:**

i. (a) $\begin{matrix} & \textit{Calorise} & \textit{Proteins}\\ \textit{Man} & 2400 & 45\\ \textit{Woman} & 1900 & 55\\ \textit{Children} & 1800 & 33\end{matrix}$

**Solution:**

Let F be the matrix representing the number of family members and R be the matrix representing the requirement of calories and proteins for each person. Then

$$F=\begin{matrix}\textit{Family A}\\ \textit{Family B}\end{matrix}\overset{\begin{matrix}\textit{Men} & \textit{Women} & \textit{Children}\end{matrix}}{\begin{bmatrix}4 & 4 & 4\\ 2 & 2 & 2\end{bmatrix}}$$

$$R=\begin{matrix}\textit{Man}\\ \textit{Woman}\\ \textit{Children}\end{matrix}\overset{\begin{matrix}\textit{Calorise} & \textit{Proteins}\end{matrix}}{\begin{bmatrix}2400 & 45\\ 1900 & 55\\ 1800 & 33\end{bmatrix}}$$

ii. (b) 24400

**Solution:**

The requirement of calories and proteins for each of the two families is given by the product matrix FR.

$$FR=\begin{bmatrix}4 & 4 & 4\\ 2 & 2 & 2\end{bmatrix}\begin{bmatrix}2400 & 45\\ 1900 & 55\\ 1800 & 33\end{bmatrix}$$

$$=\begin{bmatrix}4(2400+1900+1800) & 4(45+55+33)\\ 2(2400+1900+1800) & 2(45+55+33)\end{bmatrix}$$

$$FR=\overset{\begin{matrix}\textit{Calories} & \textit{Proteins}\end{matrix}}{\begin{bmatrix}24400 & 532\\ 12200 & 266\end{bmatrix}}\begin{matrix}\textit{Family A}\\ \textit{Family B}\end{matrix}$$

iii. (c) 266 grams

iv. (c) A + B

Solution:

Since, AB = B ....(i)

BA = A ....(ii)

$\therefore A^2 + B^2 = A\times A + B\times B$

= A(BA) + B(AB)

= (AB)A + (BA)B

= A + B

v. (a) m = q

Solution:

$A=(a_{ij})_{m\times n}, B=(b_{ij})_{n\times p}, C=(c_{ij})_{p\times q}$

$BC=(b_{ij})_{n\times p}\times(c_{ij})_{p\times q}=(d_{ij})_{n\times q}$

$(BC)A=(d_{ij})_{n\times q}\times(a_{ij})_{m\times n}$

Hence, (BC)A is possible only when $m = q$.

# Determinants 4

## Top Terms

1. To every square matrix A = $[a_{ij}]$, a unique number (real or complex) called the determinant of the square matrix A can be associated. The determinant of matrix A is denoted by det(A) or |A| or Δ.
2. Only square matrices can have determinants.
3. A determinant can be thought of as a function which associates each square matrix to a unique number (real or complex).

   f : M → K is defined by f(A) = k, where A ∈ M set of square matrices and k ∈ K set of numbers (real or complex).
4. Let A = [a] be a matrix of order 1, then the determinant of A is defined to be equal to a.
5. Determinant of order 2

   If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$
6. Determinant of order 3

   If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

   $$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
7. Minor of an element $a_{ij}$ Of A determinant is the determinant obtained by deleting its $i^{th}$ row and $j^{th}$ column in which the element $a_{ij}$ lies. Minor of an element $a_{ij}$ is denoted by $M_{ij}$. The order of minor matrix is (n -1).
8. Cofactor of an element $a_{ij}$ denoted by $A_{ij}$ is defined by

   $A_{ij} = (-1)^{i+j} M_{ij}$, where $M_{ij}$ is the minor of $a_{ij}$.
9. The adjoint of a square matrix A = $[a_{ij}]$ is the transpose of the cofactor matrix $[A_{ij}]_{n\times n}$.
10. A square matrix A is said to be singular if |A| = 0.
11. A square matrix A is said to be non-singular if |A| ≠ 0.
12. If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.
13. The determinant of the product of the matrices is equal to the product of the respective determinants, i.e., |AB| = |A||B|, where A and B are square matrices of the same order.
14. A square matrix A is invertible, i.e., its inverse exists if and only if A is a non-singular matrix. Inverse of matrix A (if it exists) is given by

   $$A^{-1}=\frac{1}{|A|}(adj\ A)$$

15. A system of equations is said to be consistent if its solution (one or more) exists.
16. A system of equations is said to be inconsistent if its solution does not exist.

**Notations to evaluate determinants:**

i. $R_i$ to denote the $i^{th}$ row.

ii. $R_i \leftrightarrow R_j$ to denote the interchange of the $i^{th}$ and $j^{th}$ rows.

iii. $R_i \leftrightarrow R_j + \lambda R_j$ to denote the addition of λ times the elements of the $j^{th}$ row to the corresponding elements of the $i^{th}$ row.

iv. $R_i(\lambda)$ to denote the multiplication of all elements of the $i^{th}$ row by λ.

v. Similar notations are used to denote column operations.

## Top Concepts

1. A determinant can be expanded along any of its rows (or columns). For easier calculations, it must be expanded along the row (or column) containing maximum zeroes.
2. **Property 1:** Value of the determinant remains unchanged if its rows and columns are interchanged. If A is a square matrix, then det(A) = det(A'), where A' = transpose of A.
3. **Property 2:** If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
4. **Property 3:** If $A = [a_{ij}]$ is a square matrix of order n and B is the matrix obtained from A by multiplying each element of a row (or column) of A by a constant k, then its value gets multiplied by k. If $\Delta_1$ is the determinant obtained by applying $Ri \to kRi$ or $Ci \to kCi$ to the determinant Δ, then $\Delta_1 = k\Delta$. Thus, |B| = k|A|. This property enables removing the common factors from a given row or column.
5. If A is a square matrix of order n and k is a scalar, then $|kA| = k^n |A|$.
6. **Property 4:** If in a determinant, the elements in two rows or columns are proportional, then the value of the determinant is zero.

   Example: $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ka_1 & ka_2 & ka_3 \end{vmatrix} = 0$ (rows $R_1$ and $R_3$ are proportional)
7. **Property 5:** If the elements of a row (or column) of a determinant are expressed as the sum of two terms, then the determinant can be expressed as the sum of the two determinants.
8. **Property 6:** If to any row or column of a determinant, a multiple of another row or column is added, then the value of the determinant remains the same, i.e., the value of the determinant remains the same on applying the operation $R_i \to Ri + kRj$ or $Ci \to C_i + kC_j$.
9. **Property 7:** If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes by a minus sign only.
10. 
11. Let A be a square matrix of order n(12) such that each element in a row (column) of A is zero, then |A|= 0.
12. If A - $[a_{ij}]$ is a diagonal matrix of order n(≥2) , then $|A| a_{11}.a_{22}\ a_{33} \dots a_{nn}$.
13. If A and B are square matrices of the same order, then |AB| = |A|.|B|
14. If more than one operation such as $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row which is affected in one operation should not be used in another operation. A similar remark applies to column operations.
15. Because area is a positive quantity, the absolute value of the determinant is taken in case of finding the area of a triangle.
16. If the area is given, then both positive and negative values of the determinant are used for calculation.

17. The area of a triangle formed by three collinear points is zero.
18. If A is a skew-symmetric matrix of odd order, then |A| = 0.
19. The determinant of a skew symmetric matrix of even order is a perfect square.
20. The minor of an element of a determinant of order n(n ≥ 2) is a determinant of order n-1.
21. The value of determinant of a matrix A is obtained by the sum of the product of the elements of a row (or column) with its corresponding cofactors. Example: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13} + A_{13}$.
22. If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

    Example: $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$
23. Adjoint of a matrix: The adjoint of a square matrix A $[a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where $A_{ij}$ is the cofactor of the element $a_{ij}$. Adjoint of the matrix A is denoted by adj A.
24. If A is any given square matrix of order n, then A (adj A) = (adj A) A = |A| I, where I is the identity matrix of order n.
25. A square matrix A is said to be singular if |A| = 0.
26. A square matrix is invertible if and only if A is a non-singular matrix.
27. The adjoint of a symmetric matrix is also a symmetric matrix.
28. If A is a non-singular matrix of order n, then $|adj.A| = |A|^{n-1}$.
29. If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.
30. If A is a non-singular square matrix, then $adj(adj\ A) = |A|^{n-2} A$.
31. Determinants can be used to find the area of triangles whose vertices are given.
32. Determinants and matrices can also be used to solve the system of linear equations in two or three variables.
33. System of equations,

    $$a_1x+b_1y+c_1z=d_1$$
    $$a_2x+b_2y+c_2z=d_2$$
    $$a_3x+b_3y+c_3z=d_3$$

    can be written as AX = B, where

    $$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

    Then matrix $X = A^{-1} B$ gives the unique solution of the system of equations if |A| is non-zero and $A^{-1}$ exists.

## Top Formulae

1. Area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2. Determinant of a matrix $A = [a_{ij}]_{1 \times 1}$ is given by $|a_{11}| = a_{11}$
3. If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then, $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

4. Cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j}. M_i$.

5. If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $adj.A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Change Sign Interchange

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where $A_{ij}$ is a cofactor of $a_{ij}$.

6. If A and B are the square matrices of the same order, then |AB| = |A| |B|.

7. $A^{-1}=\dfrac{1}{|A|}(adj\ A)$, where $|A|\neq 0$.

8. If A and B are non-singular matrices of the same order, then adj(AB) = (adj B)(adj A) .

9. If A is an invertible square matrix, then adj $A^T$-(adj A)$^T$.

10. Let A, B and C be square matrices of the same order n. If A is a non-singular matrix, then
 (i) AB = AC $\Rightarrow$ B = C
 BA = CA $\Rightarrow$ B = C

11. If A and B are two invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$

12. If A, B and C are invertible matrices of the same order, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

13. If A is an invertible square matrix, then AT is also invertible and $(AT)^{-1} = (A^{-1})^T$

14. The inverse of an invertible symmetric matrix is a symmetric matrix.

15. If A is a non-singular matrix of order n, then adj (adj A) = $|A|^{(n-2)}A$

16. $|A^{-1}|=\dfrac{1}{|A|}$ and $(A^{-1})^{-1}=A$

17. Cramer's rule (system of two simultaneous equations with two unknowns): The solution of the system of simultaneous linear equations,

 $a_1x + b_1y = c_1$

 $a_2x + b_2y = c_2$,

 is given by:

 $x=\dfrac{D_1}{D}, y=\dfrac{D_2}{D}$, where

 $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & a_2 \end{vmatrix}, D_1=\begin{vmatrix} c_1 & b_1 \\ c_2 & a_2 \end{vmatrix}$ and $D_2=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

 provided that $D\neq 0$.

18. Cramer's rule (system of three simultaneous equations with three unknowns): The solution of the system of simultaneous linear equations,

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

is given by:

$x = \frac{D_1}{D}, y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$ where

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

provided that $D \neq 0$.

19. For a system of two simultaneous linear equations with two unknowns:
    - i. If D ≠ 0, then the given system of equations is consistent and has a unique solution, given by
    - ii. $x = \frac{D_1}{D}, y = \frac{D_2}{D}$
    - iii. If D = 0 and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.
    - iv. If D = 0 and one of $D_1$ and $D_2$ is non-zero, then the system is inconsistent.
20. For a system of three simultaneous linear equations in three unknowns:
    - i. If D ≠ 0, then the given system of equations is consistent and has a unique solution given by
    - ii. $x = \frac{D_1}{D}, y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$
    - iii. If D = 0 and $D_1 = D_2 = D_3 = 0$, then the system is consistent and has infinitely many solutions.
21. Unique solution of equation AX = B is given by $X = A^{-1}B$, where |A| ≠ 0.
22. For a square matrix A in matrix equation AX = B,
    - i. If |A| ≠ 0, then there exists a unique solution.
    - ii. If |A| = 0 and (adj A) B ≠ 0, then there exists no solution.
    - iii. If |A| = 0 and (adj A) B = 0, then the system may or may not be consistent.

**Class : 12th Maths**
**Chapter- 4 : Determinants**

# Determinants

## Minors and cofactors of a matrix

Minor of an element $a_{ij}$ in a determinant of matrix $A$ is the determinant obtained by deleting $i^{th}$ row and $j^{th}$ column and is denoted by $M_{ij}$. If $M_{ij}$ is the minor of $a_{ij}$ and cofactor of $a_{ij}$ is $A_{ij}$ given by $A_{ij} = (-1)^{i+j} M_{ij}$.

- If $A_{3\times3}$ is a matrix, then $|A| = a_{11}.A_{11} + a_{12}.A_{12} + a_{13}.A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For e.g., $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$.

e.g., if $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then $M_{11} = 4$ and $A_{11} = (-1)^{1+1}4 = 4$.

## Determinant of square matrix 'A' |A| is given by

(i) if $A = [a_{11}]_{1\times1}$, then $|A| = a_{11}$

(ii) if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ then $|A| = a_{11}a_{22} - a_{12}a_{21}$

(iii) if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$, then $|A| = a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{23}.a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

For eg. if $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A| = 2\times4 - 3\times2 = 2$

## Properties of |A|

(i) $|A|$ remains unchanged, if the rows and columns of $A$ are interchanged ie., $|A| = |A'|$

(ii) if any two rows (or columns) of $A$ are interchanged, then the sign of $|A|$ changes.

(iii) if any two rows (or columns) of $A$ are identical, then $|A| = 0$

(iv) if each element of a row (or a column) of $A$ is multiplied by $B$ (const.), then $|A|$ gets multiplied by $B$.

(v) if $A = [a_{ij}]_{3\times3}$ then $|k.A| = k^3|A|$.

(vi) if elements of a row or a column in a determinant $|A|$ can be expressed as sum of two or more elements, then $|A|$ can be expressed as $|B| + |C|$.

(vii) if $R_i \to R_i + kR_j$ or $C_i = C_i + kC_j$ in $|A|$, then the value of $|A|$ remains same

## Adjoint and inverse of a matrix

if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj. $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where $A_{ij}$ is the cofactor of $a_{ij}$.

- $A(\text{adj}.A) = (\text{adj}.A).A = |A| = I$, $A$ – square matrix of order '$n$'
- if $|A| = 0$, then A is singular. Otherwise, $A$ is non-singular.
- if $AB = BA = I$, where B is a square matrix, then B is called the inverse of $A$, $A^{-1} = B$ or $B^{-1} = A$, $(A^{-1})^{-1} = A$.

Inverse of a square matrix exists if $A$ is non-singular i.e. $|A| \neq 0$, and is given by

$$A^{-1} = \frac{1}{|A|}(\text{adj}.A)$$

## Area of triangle

if $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

For eg: if (1, 2) , (3, 4) and (–2, 5) are the vertices, then area of the triangle is

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix} = 1(4-5) - 2(3+2) + 1(15+8) = 12 \text{ sq.units.}$$

we take positive value of the determinant.

## Applications of Determinants

- if $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ then we can write $AX = B$,

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

- Unique solution of $AX = B$ is $X = A^{1}B$ , $|A| \neq 0$.
- $AX = B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix $A$ in $AX = B$, if
  (i) $|A| \neq 0$ then there exists unique solution.
  (ii) $|A| = 0$ and (adj. $A$) $B \neq 0$ ,then no solution.
  (iii) if $|A| = 0$ and (adj.$A$).$B = 0$ then system may or may not be consistent.

# Important Questions

## Multiple Choice Questions-

1. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to
   (a) 6
   (b) ±6
   (c) -6
   (d) 6, 6
2. Let A be a square matrix of order 3 × 3. Then |kA| is equal to
   (a) k |A|
   (b) $k^2|A|$
   (c) $k^3|A|$
   (d) 3k|A|
3. Which of the following is correct?
   (a) Determinant is a square matrix
   (b) Determinant is a number associated to a matrix
   (c) Determinant is a number associated to a square matrix
   (d) None of these.
4. If area of triangle is 35 sq. units with vertices (2, -6), (5, 4) and (k, 4). Then k is
   (a) 12
   (b) -2
   (c) -12, -2
   (d) 12, -2.
5. If and $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $A_{ij}$ is co-factors of $a_{ij}$, then A is given by
   (a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
   (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{33}$
   (c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
   (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$
6. Let A be a non-singular matrix of order 3 × 3. Then |adj. A| is equal to
   (a) |A|
   (b) $|A|^2$
   (c) $|A|^3$
   (d) 3|A|
7. If A is any square matrix of order 3 x 3 such that |a| = 3, then the value of |adj. A| is?
   (a) 3
   (b) $\frac{1}{3}$
   (c) 9
   (d) 27
8. If A is an invertible matrix of order 2, then det (A-1) is equal to
   (a) det (A)
   (b) $\frac{1}{\det(A)}$
   (c) 1
   (d) 0
9. If a, b, c are in A.P., then determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is:
   (a) 0
   (b) 1
   (c) $x$
   (d) $2x$
10. If x, y, z are non-zero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is
   (a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$
   (b) $xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$
   (c) $\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$
   (d) $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

## Very Short Questions:

1. Find the co-factor of the element a23 of the determinant:

$$\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

2. If A and B are invertible matrices of order 3, |A| = 2 and $|(AB)^{-1}| = -\frac{1}{6}$ Find |B|.

3. Check whether (l + m + n) is a factor of the determinant $\begin{vmatrix} l+m & m+n & n+1 \\ n & l & m \\ 2 & 2 & 2 \end{vmatrix}$ or not. Given reason.

4. If A is a square matrix of order 3, with |A| = 9, then write the value of |2 . adj. A|.

5. If A and B are square matrices of the same order 3, such that |A| = 2 and AB = 2I, write the value of IBI.

6. A is a square matrix with |A| = 4. Then find the value of |A. (adj. A)|.

7. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write:

   (i) the minor of the element a23.

   (ii) the co-factor of the element a32.

8. Find the adjoint of the matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$

9. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ compute $A^{-1}$ and show that $2A^{-1} = 9I - A$.

10. For what value of 'x', the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?

## Long Questions:

1. Using properties of determinants, prove the following:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$$

If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, using properties of determinants, find the value

2. Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$$

3. Using properties of determinants, prove that:

$$\begin{vmatrix} s & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$$

## Assertion and Reason Questions-

**1.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

e) Both A and R are false.

**Assertion(A):** Minor of element 6 in the matrix $\begin{bmatrix} 0 & 2 & 6 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ is 3.

**Reason (R):** Minor of an element $a_{ij}$ of a matrix is the determinant obtained by deleting its $i^{th}$ row.

**2.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

e) Both A and R are false.

**Assertion (A):** For two matrices A and B of order 3, |A|=3, |B|=−4, then |2AB| is −96.

**Reason(R):** For a matrix A of order n and a scalar k, $|kA|=k^n|A|$.

## Case Study Questions-

**1.** Raja purchases 3 pens, 2 pencils and 1 mathematics instrument box and pays ₹41 to the shopkeeper. His friends, Daya and Anil purchases 2 pens, 1 pencil, 2 instrument boxes and 2 pens, 2 pencils and 2 mathematical instrument boxes respectively. Daya and Anil pays ₹29 and ₹44 respectively. Based on the above information answer the following:

(i) The cost of one pen is:

a) ₹2

b) ₹5

c) ₹10

d) ₹15

(ii) The cost of one pen and one pencil is:

a) ₹ 5

b) ₹10

c) ₹15

d) ₹17

(iii) The cost of one pen and one mathematical instrument box is:

a) ₹ 7

b) ₹10

c) ₹15

d) ₹18

(iv) The cost of one pencil and one mathematical instrumental box is:

a) ₹ 5

b) ₹10

c) ₹15

d) ₹20

(v) The cost of one pen, one pencil and one mathematical instrumental box is:

a) ₹ 10

b) ₹15

c) ₹22

d) ₹25

**2.** The management committee of a residential colony decided to award some of its members (say x) for honesty, some (sayy) for helping others and some others (sayz) for supervising the workers to kept the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. The sum of the number of awardees for honesty and supervision is twice the number of awardees for helping.


(i) Value of $x + y + z$ is

(a) 3

(b) 5

(c) 7

(d) 12

(ii) Value of $x - 2y$ is

(a) z

(b) -z

(c) 2z

(d) -2z

(iii) The value of z is

(a) 3

(b) 4

(c) 5

(d) 6

(iv) The value of $x + 2y$ is

(a) 9

(b) 10

(c) 11

(d) 12

(v) The value of $2x + 3y + 5z$ is

(a) 40

(b) 43

(c) 50

(d) 53

# Answer Key

## Multiple Choice Questions-

1. **Answer:** (a) 6
2. **Answer:** (c) $k^3|A|$
3. **Answer:** (c) Determinant is a number associated to a square matrix
4. **Answer:** (d) 12, -2.
5. **Answer:** (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$
6. **Answer:** (b) $|A|^2$
7. **Answer:** (c) 9
8. **Answer:** (b) $\frac{1}{\det(A)}$
9. **Answer:** (a) 0
10. **Answer:** (a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

## Very Short Answer:

**1. Solution:**

Co-factor of $a_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= (-1)^5 (5 \times 2 - 1 \times 3)$

$= (-1)(10-3)$

$= (-1)(7) = -7.$

**2. Solution:**

$$\left|(AB)^{-1}\right| = -\frac{1}{6}$$

$$\Rightarrow \frac{1}{|AB|} = -\frac{1}{6}$$

$$\Rightarrow \frac{1}{|A||B|} = -\frac{1}{6}$$

$$\Rightarrow \frac{1}{2|B|} = -\frac{1}{6}$$

Hence $|B| = 3$

**3. Solution:**

Given

$$\det. = \begin{vmatrix} l+m+n & m+n+l & n+l+m \\ n & l & m \\ 2 & 2 & 2 \end{vmatrix}$$

[Applying $R_1 \to R_1 + R_2$]

$$= (l+m+n)\begin{vmatrix} 1 & 1 & 1 \\ n & l & m \\ 2 & 2 & 2 \end{vmatrix}.$$

Hence, $(l + m + n)$ is a factor of given determinant.

**4. Solution:**

$|2 - \text{adj. } A| = 2^3 \, |A|^{3-1}$

$= 8(9)^2$

$= 648.$

**5. Solution:**

We have: $AB = 2I$

$\therefore |AB| = |2I|$

$\Rightarrow |A||B| = |2I|$

$\Rightarrow 2|B| = 2(1).$

Hence, $|B| = 1.$

**6. Solution:**

$|A. (\text{adj. } A)| = |A|^n$

$= 4^n$ or 16 or 64.

**7. Solution:**

(i) $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= (5)(2) - (1)(3)$

$= 10 - 3 = 7.$

(ii) $a_{32} = (-1)^{3+2} \begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix}$

$= (-1)^5 [(5)(1) - (2)(8)]$

$= (-1)^5 (5 - 16)$

$= (-1)(-11) = 11.$

**8. Solution:**

Here $|A| = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$

Now $A_{11}$ = Co-factor of 2 = 3,

$A_{12}$ = Co-factor of – 1 = – 4,

$A_{21}$ = Co-factor of 4 = 1

and $A_{22}$ = Co-factor of 3 = 2

$\therefore$ Co-factor matrix $= \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

Hence, adj. $A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$

**9. Solution:**

(i) We have, $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$

$\therefore \quad |A|=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$

= (2) (7) – (-4) (-3)

= 14 – 12 = 2 ≠ 0.

∴ $A^{-1}$ exists and

$$A^{-1}=\frac{1}{|A|}adj\ A=\frac{1}{2}\begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$$

(ii) **R.H.S.** = 9I – A

$$=9\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$$

$$=\begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}+\begin{bmatrix}-2 & 3\\ 4 & -7\end{bmatrix}$$

$$=\begin{bmatrix}9-2 & 0+3\\ 0+4 & 9-7\end{bmatrix}=\begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$$

$$=2\times\frac{1}{2}\begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$$

= $2A^{-1}$ = **L.H.S.**

**10. Solution:**

The matrix $\begin{bmatrix}5-x & x+1\\ 2 & 4\end{bmatrix}$ is singular

⇒ $\begin{bmatrix}5-x & x+1\\ 2 & 4\end{bmatrix}=0$

⇒ 4(5 -x) – 2 (x + 1) = 0

⇒ 20 – 4x – 2x – 2 = 0

⇒ 18 – 6x = 0

⇒ 6x = 18.

Hence, x = 3.

## Long Answer:

**1. Solution:**

**L.H.S.** $\begin{vmatrix}a+b+c & a+b & a+c\\ -c & a+b & -(a+c)\\ -b & -(a+b) & a+c\end{vmatrix}$

[Operating $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$]

$$=(a+b)(a+c)\begin{vmatrix}a+b+c & 1 & 1\\ -c & 1 & -1\\ -b & -1 & 1\end{vmatrix}$$

$$=(a+b)(a+c)\begin{vmatrix}a+b+c & 1 & 2\\ -c & 1 & 0\\ -b & -1 & 0\end{vmatrix}$$

[Operating $C_3 \to C_3 + C_2$]

= (a + b) (a + c) (2) [c + b]

= 2(a + b) (b + c) (c + a) = **R.H.S.**

**2. Solution:**

We have

$$f(x)=\begin{vmatrix}a & -1 & 0\\ ax & a & -1\\ ax^2 & ax & a\end{vmatrix}$$

$$=a\begin{vmatrix}1 & -1 & 0\\ x & a & -1\\ x^2 & ax & a\end{vmatrix}$$

[Taking a common from $C_1$]

$$=a\begin{vmatrix}1 & 0 & 0\\ x & a+x & -1\\ x^2 & ax+x^2 & a\end{vmatrix}$$

[Operating $C_2 \to C_2 + C_1$]

= $a[(a + x)\, a + (ax + x^2)]$

= $a[a2 + ax + ax + x^2)]$

= $a(x^2 + 2\,ax + a^2)$

= $a(x + a)^2$

$f(2x) = a(2x + a)^2$

$f(2x) - f(x) = a[(2x + a)^2 - (x + a)^2]$

= $a[(4x^2 + 4ax + a^2) - (x^2 + 2ax + a^2)]$

= $a(3x^2 + 2ax)$

= $ax(3x + 2a)$

**3. Solution:**

**L.H.S.** $=\begin{vmatrix}1 & 1 & 1+3x\\ 1+3y & 1 & 1\\ 1 & 1+3z & 1\end{vmatrix}$

$$=\begin{vmatrix}1 & 0 & 3x\\ 1+3y & -3y & -3y\\ 1 & 3x & 0\end{vmatrix}$$

[Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$=1.\begin{vmatrix}-3y & -3y\\ 3z & 0\end{vmatrix}+3x\begin{vmatrix}1+3y & -3y\\ 1 & 3z\end{vmatrix}$$

[Expanding by $R_1$]

(0 + 9yz) + 3x(3 z + 9yz + 3y)

= 9(3xyz + xy + yz + zx) = RHS

**4. Solution:**

$$\text{L.H.S.} = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix}$$

[Operating $C_1 \to a\ C_1$]

$$= \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$$

[Operating $C1 \to C_1 + b\ C_2 + c\ C_3$]

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

[Taking $(a_2 + b_2 + c_2)$ common from $C_1$]

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

[Operating $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$]

$$= \frac{1}{a}(a^2+b^2+c^2)(1)\begin{vmatrix} c & -a-b \\ a+c & -b \end{vmatrix}$$

$$= \frac{a^2+b^2+c^2}{a}[-bc+a^2+ac+ba+bc]$$

$$= \frac{(a^2+b^2+c^2)}{a}(a)(a+b+c)$$

$$=(a+b+c)(a^2+b^2+c^2) = \textbf{R.H.S.}$$

## Case Study Answers-

**1.**

(i) (a) ₹ 2

(ii) (d) ₹ 17

(iii) (a) ₹ 7

(iv) (d) ₹20

(v) (c) ₹22

**2.**

(i) (d) 12

(ii) (b)-z

(iii) (c)5

(iv) (c) 11

(v) (b)43

## Assertion and Reason Answers-

**1.** (e) Both A and R are false.

**Solution:**

Minor of element $6 = M_{12} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-4 = -3$

∴ Given Assertion [A] is false Also we know that minor of an element $a_{ij}$ of a matrix is the determinant obtained by deleting its $i^{th}$ row and $j^{th}$ column.

∴ Given Reason (R) is also false

∴ Both Assertion [A] and Reason [R] are false Hence option (e) is the correct Answer.

**2.** (b) Both A and R are true but R is not the correct explanation of A.

**Solution:**

Here,

|2AB|=23|AB|=8|A||B|

=8×3×−4=−96

∴ Assertion [A] is true

{∵|kA|=kn|A|and|AB|=|A||B|}

Also we know that |kA|=kn|A|

for matrix A of order n.

∴ Reason (R) is true But |AB|=|A||B|is not mentioned in Reason R.

∴ Both A and R are true but R is not correct explanation of A Hence option (b) is the correct answer.

# Continuity and Differentiability 5

## Top Definitions

1. A function f(x) is said to be continuous at a point c if.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

2. A real function f is said to be continuous if it is continuous at every point in the domain of f.

3. If f and g are real-valued functions such that (f o g) is defined at c, then (f g)(x) - f(g(x)).

   If g is continuous at c and if f is continuous at g(c), then (f o g) is continuous at c.

4. A function f is differentiable at a point c if Left Hand Derivative (LHD) = Right Hand Derivative (RHD),

   *i.e.* $$\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$$

5. If a function f is differentiable at every point in its domain, then $\lim_{h \to 0} \frac{f(x+h)-f(c)}{h}$ or $\lim_{h \to 0} \frac{f(x-h)-f(c)}{-h}$ is called the derivative or differentiation of *f* at *x* and is denoted by denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

6. If LHD ≠ RHD, then the function f(x) is not differentiable at x = c.

7. Geometrical meaning of differentiability:

   The function f(x) is differentiable at a point P if there exists a unique tangent at point P. In other words, f(x) is differentiable at a point P if the curve does not have P as its corner point.

8. A function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

9. A function is said to be differentiable in an interval [a, b] if it is differentiable at every point of [a, b].

10. **Chain Rule of Differentiation:** If f is a composite function of two functions u and v such that f = v(t) and $t = u(x)$ and if both $\frac{dv}{dt}$ and $\frac{dt}{dx}$ exist, then $\frac{dv}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

11. Logarithm of a to the base b is x, i.e., $\log_b a = x$ if $b^x = a$, where b > 1 is a real number. Logarithm of a to base b is denoted by $\log_b a$.

12. Functions of the form x = f(t) and y = g(t) are parametric functions.

13. **Rolle's Theorem:** If f : [a, b] → R is continuous on [a, b] and differentiable on (a, b) such that f(a) = f(b), then there exists some c in (a, b) such that f'(c) = 0.

14. **Mean Value Theorem:** If f : [a, b] → R is continuous on [a, b] and differentiable on (a, b), then there exists some c in (a, b) such that $f'(c) = \lim_{h \to 0} \frac{f(b)-f(a)}{b-a}$.

## Top Concepts

1. A function is continuous at x = c if the function is defined at x = c and the value of the function at x = c equals the limit of the function at x = c.
2. If function f is not continuous at c, then f is discontinuous at c and c is called the point of discontinuity of f.
3. Every polynomial function is continuous.
4. The greatest integer function [x] is not continuous at the integral values of x.
5. Every rational function is continuous.

   ***Algebra of continuous functions:***

   i. Let f and g be two real functions continuous at a real number c, then f + g is continuous at x = c.

   ii. f - g is continuous at x = c.

   iii. f . g is continuous at x = c.

   iv. $\left(\frac{f}{g}\right)$ is continuous at x = c, [provided g(c) ≠ 0].

   v. kf is continuous at x = c, where k is a constant.
6. Consider the following functions:

   i. Constant function

   ii. Identity function

   iii. Polynomial function

   iv. Modulus function

   v. Exponential function

   vi. Sine and cosine functions

   ***The above functions are continuous everywhere.***
7. Consider the following functions:

   i. Logarithmic function

   ii. Rational function

   iii. Tangent, cotangent, secant and cosecant functions

   ***The above functions are continuous in their domains.***
8. If f is a continuous function, then |f| and $\frac{1}{f}$ are continuous in their domains.
9. Inverse functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\text{cosec}^{-1}x$ and $\sec^{-1}x$ are continuous functions on their respective domains.
10. The derivative of a function f with respect to x is f′(x) which is given by $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
11. If a function f is differentiable at a point c, then it is also continuous at that point.
12. Every differentiable function is continuous, but the converse is not true.
13. Every polynomial function is differentiable at each x ∈ R.
14. Every constant function is differentiable at each x ∈ R.
15. The chain rule is used to differentiate composites of functions.
16. The derivative of an even function is an odd function and that of an odd function is an even function.
17. **Algebra of Derivatives**

   If *u* and *v* are two functions which are differentiable, then

   i. $(u\pm v)'=u'\pm v'$ (Sum and Difference Formula)

   ii. $(uv)'=u'v+uv'$ (Leibnitz rule or Product rule)

   iii. $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}, v\neq 0,$ (Quotient rule)

18. **Implicit Functions**

If it is not possible to separate the variables x and y, then the function f is known as an implicit function.

19. **Exponential function:** A function of the form $y = f(x) = b^x$, where base b > 1.

1. Domain of the exponential function is R, the set of all real numbers.
2. The point (0, 1) is always on the graph of the exponential function.
3. The exponential function is ever increasing.

20. The exponential function is differentiable at each $x \in R$ .
21. Properties of logarithmic functions:

i. Domain of log function is $R^+$.

ii. The log function is ever increasing.

iii. For 'x' very near to zero, the value of log x can be made lesser than any given real number.

22. Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both f(x) and u(x) need to be positive.
23. To find the derivative of a product of a number of functions or a quotient of a number of functions, take the logarithm of both sides first and then differentiate.
24. **Logarithmic Differentiation**

$y = a^x$

Taking logarithm on both sides

$\log y = \log a^x$.

Using the property of logarithms

log y = x log a

Now differentiating the implicit function:

$$\frac{1}{y}.\frac{dy}{dx} = \log a$$

$$\frac{dy}{dx} = y \log a = a^x \log a$$

25. The logarithmic function is differentiable at each point in its domain.
26. Trigonometric and inverse-trigonometric functions are differentiable in their respective domains.
27. The sum, difference, product and quotient of two differentiable functions are differentiable.
28. The composition of a differentiable function is a differentiable function.
29. A relation between variables x and y expressed in the form x = f(t) and y = g(t) is the parametric form with t as the parameter. Parametric equation of parabola $y^2 = 4ax$ is $x = at^2$, $y = 2at$.
30. Differentiation of an infinite series: If f(x) is a function of an infinite series, then to differentiate the function f(x), use the fact that an infinite series remains unaltered even after the deletion of a term.
31. **Parametric Differentiation:**

Differentiation of the functions of the form x = f(t) and y = g(t):

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

32. Let u = f(x) and v = g(x) be two functions of x. Hence, to find the derivative of f(x) with respect g(x), we use the following formula:

$$\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

33. If y = f(x) and $\frac{dy}{dx}=f'(x)$ and if f'(x) is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$ or f"(x) is the second order derivative of y with respect to x.

34. If x = f(t) and y = g(t), then

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left\{\frac{g'(t)}{f'(t)}\right\}$$

or $$\frac{d^2y}{dx^2}=\frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\}\cdot\frac{dt}{dx}$$

or $$\frac{d^2y}{dx^2}=\frac{f'(t)\,g''(t)=g'(t)\,f''(t)}{\{f'(t)\}^3}$$

## Top Formulae

1. Derivative of a function at a point

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

2. Properties of logarithms

log(*xy*) = log *x* + log *y*

$$\log\left(\frac{x}{y}\right)=\log x-\log y$$

$$\log(x^y)=y\log x$$

$$\log_a x=\frac{\log_b x}{\log_b a}$$

3. Derivatives of Functions

$$\frac{d}{dx}x^n=nx^{n-1}$$

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\cot x)=-\mathrm{cosec}^2 x$$

$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = \operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}, a > 0, a \neq 1$$

$$\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \text{ if } |x| > 1$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}, \text{ if } |x| > 1$$

$$\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} = \begin{cases} -\frac{2}{1+x^2}, & x > 1 \\ \frac{2}{1+x^2}, & -1 < x < 1 \\ -\frac{2}{1+x^2}, & x < -1 \end{cases}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\} = \begin{cases} \frac{2}{1+x^2}, & x > 0 \\ \frac{-2}{1+x^2}, & x < 0 \end{cases}$$

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\} = \begin{cases} \frac{2}{1+x^2}, & x < -1 \text{ or } x > 1 \\ \frac{-2}{1+x^2}, & -1 < x < 1 \end{cases}$$

$$\frac{d}{dx}\left\{\sin^{-1}\left(3x-4x^3\right)\right\} = \begin{cases} -\frac{3}{\sqrt{1-x^2}}, & \frac{1}{2} < x < 1, -1 < x < -\frac{1}{2} \\ \frac{3}{\sqrt{1-x^2}}, & -\frac{1}{2} < x < \frac{1}{2} \end{cases}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(4x^3-3x\right)\right\} = \begin{cases} -\frac{3}{\sqrt{1-x^2}}, & -\frac{1}{2} < x < 1 \\ \frac{3}{\sqrt{1-x^2}}, & -\frac{1}{2} < x < \frac{1}{2} \text{ or } -1 < x < -\frac{1}{2} \end{cases}$$

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right\}=\begin{cases}\frac{3}{1+x^2}, x<-\frac{1}{\sqrt{3}} \text{ or } x>\frac{1}{\sqrt{3}}\\ \frac{3}{1+x^2}, x-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\end{cases}$$

$$\frac{d}{dx}\left[\sin\left(\sin^{-1}x\right)\right]=1, \text{if } -1<x<1$$

$$\frac{d}{dx}\left[\cos\left(\cos^{-1}x\right)\right]=1, \text{if } -1<x<1$$

$$\frac{d}{dx}\left[\tan\left(\tan^{-1}x\right)\right]=1, \text{for all } x\in R$$

$$\frac{d}{dx}\left[\text{cosec}\left(\text{cosec}^{-1}x\right)\right]=1, \text{for all } x\in R-(-1,1)$$

$$\frac{d}{dx}\left[\sec\left(\sec^{-1}x\right)\right]=1, \text{for all } x\in R-(-1,1)$$

$$\frac{d}{dx}\left[\cot\left(\cot^{-1}x\right)\right]=1, \text{for all } x\in R$$

$$\frac{d}{dx}\left[\sin^{-1}(\sin x)\right]=\begin{cases}-1, -\frac{3\pi}{2}<x<-\frac{\pi}{2}\\ 1, -\frac{\pi}{2}<x<\frac{\pi}{2}\\ -1, \frac{\pi}{2}<x<\frac{3\pi}{2}\\ 1, \frac{3\pi}{2}<x<\frac{5\pi}{2}\end{cases}$$

$$\frac{d}{dx}\left[\cos^{-1}(\cos x)\right]=\begin{cases}1, 0<x<\pi\\ -1, \pi<x<2\pi\end{cases}$$

$$\frac{d}{dx}\left[\tan^{-1}(\tan x)\right]=\left\{1, n\pi-\frac{\pi}{2}<x<\frac{\pi}{2}+n\pi, n\in Z\right.$$

$$\frac{d}{dx}\left[\text{cosec}^{-1}(\text{cosec}\, x)\right]=\begin{cases}1, -\frac{\pi}{2}<x<0 \text{ or } 0<x<\frac{\pi}{2}\\ -1, -\frac{\pi}{2}<x<\pi \text{ or } 0<\pi<x<\frac{3\pi}{2}\end{cases}$$

$$\frac{d}{dx}\left[\sec^{-1}(\sec x)\right]=\begin{cases}1, 0<x<\frac{\pi}{2} \text{ or } \frac{\pi}{2}<x<\pi\\ -1, \pi<x<\frac{3\pi}{2} \text{ or } \frac{3\pi}{2}<x<2\pi\end{cases}$$

$$\frac{d}{dx}\left[\cot^{-1}(\cot x)\right]=1, (n-1)\pi<x<x<n\pi, n\in Z$$

4. Differentiation of constant functions
   1. Differentiation of a constant function is zero, i.e.

      $$\frac{d}{dx}(c)=0$$

   2. If f(x) is a differentiable function and c is a constant, then cf(x) is a differentiable function such that

      $$\frac{d}{dx}(cf(x))=c\frac{d}{dx}(f(x))$$

5. Some useful results in finding derivatives

1. $\sin 2x = 2\sin x \cos x$
2. $\sin 2x = 2\cos^2 x - 1$
3. $\cos 2x = 1 - 2\sin^2 x$
4. $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$
5. $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$
6. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$
7. $\sin 3x = 3\sin x - 4\sin^3 x$
8. $\cos 3x = 4\cos^3 x - 3\cos x$
9. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$
10. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$
11. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$
12. $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\dfrac{x \pm y}{1 \mp xy}\right)$
13. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, if $-1 \le x \le 1$
14. $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$, for all $x \in R$
15. $\sec^{-1} x + cosec^{-1} x = \dfrac{\pi}{2}$, if $x \in (-\infty, -1] \cup [1, \infty)$
16. $\sin^{-1}(-x) = -\sin^{-1} x$, for $x \in [-1, 1]$
17. $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for $x \in [-1, 1]$
18. $\tan^{-1}(-x) = -\tan^{-1} x$, for $x \in R$
19. $\sin^{-1} x = \mathrm{cosec}^{-1}\left(\dfrac{1}{x}\right)$ if $x \in (-\infty, 1] \cup [1, \infty)$
20. $\cos^{-1} x = \sec^{-1}\left(\dfrac{1}{x}\right)$ if $x \in (-\infty, 1] \cup [1, \infty)$
21. $\tan^{-1} x = \begin{cases} \cot^{-1}\left(\dfrac{1}{x}\right), \text{ if } x > 0 \\ -\pi + \cot^{-1}\left(\dfrac{1}{x}\right), \text{ if } x < 0 \end{cases}$
22. $\sin^{-1}(\sin\theta) = \theta$, if $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$
23. $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$

24. $\tan^{-1}(\tan\theta)=\theta, \textit{if} -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

25. $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta, \text{if} -\frac{\pi}{2}<\theta<\frac{\pi}{2}, \theta\neq 0$

26. $\sec^{-1}(\sec\theta)=\theta, \text{if } 0<\theta<\pi, \theta\neq\frac{\pi}{2}$

27. $\cot^{-1}(\cot\theta)=\theta \text{ if } 0<\theta<\pi$

6. Substitutions useful in finding derivatives

| If the expression is | then substitute |
|---|---|
| 1. $a^2 + x^2$ | $x = a \tan\theta$ or a $\cot\theta$ |
| 2. $a^2 - x^2$ | $x = a \sin\theta$ or a $\cos\theta$ |
| 3. $x^2 - a^2$ | $x = a \sec\theta$ or a $\text{cosec}\,\theta$ |
| 4. $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$ | $x = a \cos 2\theta$ |
| 5. $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$ | $x^2 = a^2 \cos 2\theta$ |

**Class : 12th Maths**
**Chapter- 5 : Continuity and Differentiability**

# Continuity and Differentiability

### Continuous Function

Suppose $f$ is a real function on a subset of the real numbers and let '$c$' be a point in the domain of $f$. Then $f$ is continuous at $c$ if $\lim_{x\to c} f(x)=f(c)$

A real function $f$ is said to be continuous if it is continuous at every point in the domain of $f$. For eg: The function $f(x)=\frac{1}{x}, x\neq 0$ is continuous

Let 'C' be any non-zero real number, then $\lim_{x\to c} f(x) = \lim_{x\to c}\frac{1}{x}=\frac{1}{c}$ . For $c=0, f(c)=\frac{1}{c}$ So $\lim_{x\to c} f(x)=f(c)$ and hence $f$ is continuous at every point in the domain of $f$.

### Derivatives of functions in parametric form

Let $x=f(t)$, $y=g(t)$ be two functions of parameter '$t$'

Then, $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ or $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\left(\frac{dx}{dt}\neq 0\right)$

Jhus, $\frac{dy}{dx}=\frac{g'(t)}{f'(t)}$ [provided $f'(t)\neq 0$]

For $eg$ : if $x=a\cos\theta$, $y=a\sin\theta$ then $\frac{dx}{d\theta}=-a\sin\theta$ and $\frac{dy}{d\theta}=a\cos\theta$, and so $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\frac{a\cos\theta}{a\sin\theta}=-\cot\theta.$

### Differentiability

Suppose $f$ is a real function and $c$ is a point in its domain. The derivative of $f$ at $c$ is $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

### Algebra of continuous functions

Suppose $f$ and $g$ are two real functions continuous at a real number $c$, then, $f+g, f-g, f.g$ and $\frac{f}{g}$ are continuous at $x=c\,(g(c)\neq 0)$.

### Second order derivative

Let $y=f(x)$ then $\frac{dy}{dx}=f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)$ i.e., $\frac{d^2y}{dx^2}=f''(x)$ is the second order derivative of $y$ w.r.t. $x$.

For $eg$ : if $y=3x^2+2$, then $y'=6x$ and $y''=6$.

### Rolle's theorem

if $f:[a,b]\to R$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Such that $f(a)=f(b)$, then $\exists$ some c in $(a, b)$ s.t. $f'(c)=0$.

### Mean Value theorem

if $f:[a,b]\to R$ continuous on $[a, b]$ and differentiable on $(a, b)$.

Then $\exists$ some c in $(a, b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

e.g. Let $f(x)=x^2$ defined in the interval $[2, 4]$. Since $f(x)=x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as $f'(x)=2x$ defined in $(2, 4)$. So,

$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{16-4}{4-2}=6, c\in(2,4).$

### Chain Rule

if $f=v o u$, $t=u(x)$ and if both $\frac{dt}{dx},\frac{dv}{dt}$ exists, then $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}$

### Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{1-x^2}}$ (vi) $\frac{d}{dx}(\mathrm{cosec}^{-1}x)=-\frac{1}{x\sqrt{1-x^2}}$

(vii) $\frac{d}{dx}(e^x)=e^x$ (viii) $\frac{d}{dx}(\log x)=\frac{1}{x}$

### Logarithmic differentiation

Let $y=f(x)=[u(x)]^{v(x)}$

$\log y=v(x)\log[u(x)]$

$\frac{1}{y}=v(x)\frac{1}{u(x)}u'(x)+v'(x)\log[u(x)]$

$\frac{dy}{dx}=y\left[\frac{v(x)}{u(x)}u'(x)+v'(x)\log[u(x)]\right]$

For e.g. : Let $y=a^x$ Then $\log y=x\log a$

$\frac{1}{y}\cdot\frac{dy}{dx}=\log a$

$\frac{dy}{dx}=y\log a=a^x\log a.$

# Important Questions

## Multiple Choice questions-

1. The function

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

is continuous at x = 0, then the value of 'k' is:

(a) 3

(b) 2

(c) 1

(d) 1.5

2. The function f(x) = [x], where [x] denotes the greatest integer function, is continuous at:

(a) 4

(b) –2

(c) 1

(d) 1.5

3. The value of 'k' which makes the function defined by

$$f(x)=\begin{cases}\sin\dfrac{1}{x}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

continuous at $x$ = 0 is

(a) –8

(b) 1

(c) –1

(d) None of these.

4. Differential coefficient of sec ($\tan^{-1} x$) w.r.t. $x$ is

(a) $\dfrac{x}{\sqrt{1+x^2}}$

(b) $\dfrac{x}{1+x^2}$

(c) $x\sqrt{1+x^2}$

(d) $\dfrac{1}{\sqrt{1+x^2}}$

5. If $y = \log\left(\frac{1-x_2}{1+x_2}\right)$ then $\frac{dy}{dx}$ is equal to:

(a) $\dfrac{4x^3}{1-x^4}$

(b) $\dfrac{-4x}{1-x^4}$

(c) $\dfrac{1}{4-x^4}$

(d) $\dfrac{-4x^3}{1-x^4}$

6. If $y=\sqrt{\sin x+y}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{\cos x}{2y-1}$

(b) $\dfrac{\cos x}{1-2y}$

(c) $\dfrac{\sin x}{1-2y}$

(d) $\dfrac{\sin x}{2y-1}$

7. If u = sin-1 $\left(\frac{2x}{1+x_2}\right)$ and u = tan-1 $\left(\frac{2x}{1-x_2}\right)$ then $\frac{dy}{dx}$ is

(a) 12

(b) x

(c) $\frac{1-x^2}{1+x^2}$

(d) 1

8. If $x = t^2$, $y = t^3$, then $\frac{d^2y}{dx^2}$ is

(a) $\frac{3}{2}$

(b) $\frac{3}{4t}$

(c) $\frac{3}{2t}$

(d) $\frac{3t}{2}$

9. The value of 'c' in Rolle's Theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1

(b) -1

(c) $\frac{3}{2}$

(d) $\frac{1}{3}$

10. The value of 'c' in Mean Value Theorem for the function $f(x) = x(x-2)$, $x \in [1, 2]$ is

(a) $\frac{3}{2}$

(b) $\frac{2}{3}$

(c) $\frac{1}{2}$

(d) $\frac{3}{4}$

## Very Short Questions:

1. If y = log (cos ex), then find $\frac{dy}{dx}$
2. Differentiate cos {sin (x)2} w.r.t. x.
3. Differentiate $\sin^2(x^2)$ w.r.t. $x^2$.
4. Find $\frac{dy}{dx}$, if y + siny = cos or.
5. If $y=\sin^{-1}\left(6x\sqrt{1-9x^2}\right), -\frac{1}{3\sqrt{2}}<x<\frac{1}{3\sqrt{2}}$ then find $\frac{dy}{dx}$.
6. Is it true that x = $e^{\log x}$ for all real x?
7. Differentiate the following w.r.t. x : $3^{x+2}$.
8. Differentiate log (1 + θ) w.r.t. $\sin^{-1}\theta$.
9. If y = $x^x$, find $\frac{dy}{dx}$.
10. If $y=\sqrt{2^x+\sqrt{2^x+\sqrt{2^x+......+\infty}}}$ then prove that: $(2y-1)\frac{dy}{dx}=2^x\log 2$.

## Short Questions:

1. Discuss the continuity of the function: f(x) = |x| at x = 0.
2. If f(x) = x + 1, find $\frac{d}{dx}$ (fof)(x).
3. Differentiate $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$ with respect to x.
4. Differentiate: $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.
5. Write the integrating factor of the differential equation:
   $(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$.
6. Find $\frac{dy}{dx}$ if $y=\sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$
7. Find $\frac{dy}{dx}$ if $y=\sin^{-1}\left[\frac{6x-4\sqrt{1-4x^2}}{5}\right]$
8. If $y=\left\{x+\sqrt{x^2+a^2}\right\}^n$, prove that $\frac{dy}{dx}=\frac{ny}{\sqrt{x^2+a^2}}$

## Long Questions:

1. Find the value of 'a' for which the function 'f' defined as:

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1), & x\le 0 \\ \frac{\tan x-\sin x}{x^3}, & x>0 \end{cases}$$

is continuous at $x = 0$

2. Find the values of 'p' and 'q' for which:

$$\begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, \text{ if } x<\frac{\pi}{2} \\ p, \quad \text{if } x=\frac{\pi}{2} \\ \frac{q(1-\sin x)}{(\pi-2x)^2}, \text{ if } x>\frac{\pi}{2} \end{cases}$$

is continuous at $x = 2$

3. Find the value of 'k' for which

$$f(x)=\begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0 \\ \frac{2x+1}{x-1}, & \text{if } 0\le x<1 \end{cases}$$

is continuous at $x = 0$

4. For what values of 'a' and 'b\ the function 'f' defined as:

$$f(x)=\begin{cases} 3ax+b & \text{if } x<1 \\ 11 & \text{if } x=1 \\ 5ax-2b & \text{if } x>1 \end{cases}$$

is continuous at $x = 1$.

## Assertion and Reason Questions:

**1.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

**Assertion(A):** $f(x)=\begin{cases} |x|+\sqrt{x-|x|}, & x\ge 0 \\ \sin x & x<0 \end{cases}$

is continuous at $x = 0$.

**Reason (R):** Both $h(x)$ =

$x^2$, $g(x)=\begin{cases} \sin\left(\frac{1}{x}\right), & x\ne 0 \\ 0 & x=0 \end{cases}$

are continuous at x = 0.

**2.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

e) Both A and R are false.

**Assertion (A):** The function $f(x)=\begin{cases}|x|+\sqrt{x-|x|}, & x\geq 0\\ \sin x & x<0\end{cases}$ is continuous everywhere.

**Reason (R):** $f(x)$ is periodic function.

## Case Study Questions-

**1.** If a relation between x and y is such that y cannot be expressed in terms of x, then y is called an implicit function of x. When a given relation expresses y as an implicit function of x and we want to find $\frac{dy}{dx}$, then we differentiate every term of the given relation w.r.t. x, remembering that a tenn in y is first differentiated w.r.t. y and then multiplied by $\frac{dy}{dx}$.

Based on the above information, find the value of $\frac{dy}{dx}$ in each of the following questions.

i. $x^3+x^2y+xy^2+y^3=81$

(a) $\frac{(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

(b) $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

(c) $\frac{(3x^2+2xy-y^2)}{x^2+2xy+3y^2}$

(d) $\frac{3x^2+xy+y^2}{x^2+xy+3y^2}$

ii. $x^y=e^{x-y}$

(a) $\frac{x-y}{(1+\log x)}$

(b) $\frac{x+y}{(1+\log x)}$

(c) $\frac{x-y}{x(1+\log x)}$

(d) $\frac{x+y}{x(1+\log x)}$

iii. $e^{\sin y}=xy$

(a) $\frac{-y}{x(y\cos y-1)}$

(b) $\frac{y}{y\cos y-1}$

(c) $\frac{y}{y\cos y+1}$

(d) $\frac{y}{x(y\cos y-1)}$

iv. $\sin^2 x+\cos^2 y=1$

(a) $\frac{\sin 2y}{\sin 2x}$

(b) $-\frac{\sin 2x}{\sin 2y}$

(c) $-\frac{\sin 2y}{\sin 2x}$

(d) $\frac{\sin 2x}{\sin 2y}$

v. $y=\left(\sqrt{x}\right)^{\sqrt{x}^{\sqrt{x}\ldots\infty}}$

(a) $\frac{-y^2}{x(2-y\log x)}$

(b) $\frac{y^2}{2-y\log x}$

(c) $\frac{y^2}{x(2+y\log x)}$

(d) $\frac{y^2}{x(2-y\log x)}$

**1.** If y = f(u) is a differentiable function of u and u = g(x) is a differentiable function of x, then y = f[g(x)] is a differentiable function of x and $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$. This rule is also known as CHAIN RULE.

Based on the above information, find the derivative of functions w.r.t. x in the following questions.

i. $\cos\sqrt{x}$

(a) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

(b) $\frac{\sin\sqrt{x}}{2\sqrt{x}}$

(c) $\sin\sqrt{x}$

(d) $-\sin\sqrt{x}$

ii. $7^{x+\frac{1}{x}}$

(a) $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

(b) $\left(\frac{x^2+1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

(c) $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x-\frac{1}{x}}\cdot\log 7$

(d) $\left(\frac{x^2+1}{x^2}\right)\cdot 7^{x-\frac{1}{x}}\cdot\log 7$

iii. $\sqrt{\frac{1-\cos x}{1+\cos x}}$

(a) $\frac{1}{2}\sec^2\frac{x}{2}$

(b) $-\frac{1}{2}\sec^2\frac{x}{2}$

(c) $\sec^2\frac{x}{2}$

(d) $-\sec^2\frac{x}{2}$

iv. $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(a) $\frac{-1}{x^2+b^2}+\frac{1}{x^2+a^2}$

(b) $\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$

(c) $\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}$

(d) None of these

v. $\sec^{-1}x+\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

(a) $\frac{2}{\sqrt{x^2-1}}$

(b) $\frac{-2}{\sqrt{x^2-1}}$

(c) $\frac{1}{|x|\sqrt{x^2-1}}$

(d) $\frac{2}{|x|\sqrt{x^2-1}}$

# Answer Key

## Multiple Choice Questions-

1. **Answer:** (b) 2
2. **Answer:** (d) 1.5.
3. **Answer:** (d) None of these.
4. **Answer:** (a) $\frac{x}{\sqrt{1+x^2}}$
5. **Answer:** (b) $\frac{-4x}{1-x^4}$
6. **Answer:** (a) $\frac{\cos x}{2y-1}$
7. **Answer:** (d) 1
8. **Answer:** (b) $\frac{3}{4t}$
9. **Answer:** (a) 1
10. **Answer:** (a) $\frac{3}{2}$

## Very Short Answer:

1. **Solution:**

We have: y = log (cos $e^x$)

$\therefore \frac{dy}{dx}=\frac{1}{\cos e^x}\left(-\sin e^x\right)\cdot e^x$

= – $e^x$ tan $e^x$

2. **Solution:**

Let y = cos {sin $(x)^2$}.

$\therefore \frac{dy}{dx}$ = – sin {sin $(x)^2$}. $\frac{dy}{dx}$ {sin $(x)^2$}

$= -\sin\{\sin(x)^2\}.\cos(x)^2 \frac{dy}{dx}(x^2)$

$= -\sin\{\sin(x)^2\}.\cos(x)^2 2x$

$= -2x\cos(x)^2 \sin\{\sin(x)^2\}$.

**3. Solution:**

Let $y = \sin^2(x^2)$.

$\therefore \frac{dy}{dx} = 2\sin(x^2)\cos(x^2) = \sin(2x^2)$.

**4. Solution:**

We have: $y + \sin y = \cos x$.

Differentiating w.r,t. x, we get:

$\frac{dy}{dx} + \cos y.\frac{dy}{dx} = -\sin x$

$(1+\cos y)\frac{dy}{dx} = -\sin x$

Hence, $\frac{dy}{dx} = -\frac{\sin x}{1+\cos y}$

where $y \neq (2n+1)\pi, n \in Z$.

**5. Solution:**

Here $y=\sin^{-1}\left(6x\sqrt{1-9x^2}\right)$

Put $3x = \sin\theta$.

$y = \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta) = 2\theta$

$= 2\sin^{-1} 3x$

$$\frac{dy}{dx}=\frac{6}{\sqrt{1-9x^2}}$$

**6. Solution:**

The given equation is $x = e^{\log x}$

This is not true for non-positive real numbers.

[ ∵ Domain of log function is R+]

Now, let $y = e^{\log x}$

If $y > 0$, taking logs.,

$\log y = \log(e^{\log x}) = \log x.\log e$

$= \log x . 1 = \log x$

$\Rightarrow y = x$.

Hence, $x = e^{\log x}$ is true only for positive values of x.

**7. Solution:**

Let $y = 3^{x+2}$.

$\frac{dy}{dx} = {}^{3x+2}.\log 3.\frac{d}{dx}(x+2)$

$= 3^{x+2}.\log 3.(1+0)$

$= 3^{x+2}.\log 3 = \log 3\,(3^{x+2})$.

**8. Solution:**

Let $y = \log(1+\theta)$ and $u = \sin^{-1}\theta$.

$$\therefore \frac{dy}{d\theta}=\frac{1}{1+\theta} \text{ and } \frac{du}{d\theta}=\frac{1}{\sqrt{1-\theta^2}}.$$

$$\therefore \frac{dy}{du}=\frac{dy/d\theta}{du/d\theta}$$

$$=\frac{\frac{1}{1+\theta}}{\frac{1}{\sqrt{1-\theta^2}}}=\sqrt{\frac{1-\theta}{1+\theta}}$$

**9. Solution:**

Here $y = x^x$ ...(1)

Taking logs., $\log y = \log x^x$

$\Rightarrow \log y = x\log x$.

Differentiating w.r.t. x, we get:

$\frac{1}{y}\cdot\frac{dy}{dx} = x\,1x + \log x$. (1)

$= 1 + \log x$.

Hence, $\frac{dy}{dx} = y(1+\log x)\,dx$

$= x^x(1+\log x)$. [Using (1)]

**10. Solution:**

The given series can be written as:

$$y=\sqrt{2^x+y}$$

Squaring, $y^2 = 2^x + y$

$\Rightarrow y^2 - y = 2^x$.

Diff. w.r.t. x, $(2y-1)\frac{dy}{dx} = 2^x\log 2$.

## Short Answer:

**1. Solution:**

By definition, $f(x)=\begin{cases}-x, \text{ if } x<0\\ x, \text{ if } x\geq 0\end{cases}$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-x)$$

$$=\lim_{h\to 0}-(-(0-h))$$

$$=\lim_{h\to 0}-(h)=0$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x)$$

$$=\lim_{h\to 0}-(0+h)$$

$$=\lim_{h\to 0}(h)=0$$

Also $f(0) = 0$

Thus $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$

[∵ Each = 0]

Hence 'f' is continuous at x = 0.

**2. Solution:**

We have : f(x) = x + 1 ...(1)

∴ fof(x) = f (f(x)) =f(x)+ 1

= (x + 1) + 1 = x + 2.

∴ $\frac{d}{dx}$ (fof)(x).) = $\frac{d}{dx}$ (x + 2) = 1 + 0 = 1.

**3. Solution:**

Let $y=\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$

[Dividing num. & denom. by cos $x$]

$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-x\right)\right)=\frac{\pi}{4}-x$

Differentiating (1) w.r.t. $x$,

$\Rightarrow \quad \frac{dy}{dx}=-1$

**4. Solution:**

Let $y=\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$=\tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$

$=\tan^{-1}\left(\cot\frac{x}{2}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)=\frac{\pi}{2}-\frac{x}{2}$

$\therefore \frac{dy}{dx}=0-\frac{1}{2}=-\frac{1}{2}.$

**5. Solution:**

The given differential equation is:

$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$

$\Rightarrow \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$ Linear Equation

$\therefore$ I.F $=e^{\int\frac{1}{1+y^2}dx}=e^{\tan^{-1}y}$

**6. Solution:**

We have: $y=\sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$

$=\sin^{-1}\left(\frac{5}{13}x+\frac{12}{13}\sqrt{1-x^2}\right)$

$=\sin^{-1}\left(x\sqrt{1-\left(\frac{12}{13}\right)^2}+\sqrt{1-x^2}\cdot\frac{12}{13}\right)$

(Note this step)

$=\sin^{-1}x+\sin^{-1}\frac{12}{13}$

$\left[\therefore \sin^{-1}A+\sin^{-1}B=\sin^{-1}\left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right)\right]$

$\therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+0=\frac{1}{\sqrt{1-x^2}},\ |x|<1$

**7. Solution:**

We have: $y=\sin^{-1}\left(\frac{6x-4\sqrt{1-4x^2}}{5}\right)$

$=\sin^{-1}\left(\frac{6x}{5}-\frac{4}{5}\sqrt{1-4x^2}\right)$

$=\sin^{-1}\left((2x)\cdot\frac{3}{5}-\frac{4}{5}\sqrt{1-(2x)^2}\right)$

$=\sin^{-1}\left((2x)\sqrt{1-\left(\frac{4}{5}\right)^2}-\left(\frac{4}{5}\right)\sqrt{1-(2x)^2}\right)$

$=\sin^{-1}(2x)-\sin^{-1}\frac{4}{5}$

Hence, $\frac{dy}{dx}=\frac{1}{\sqrt{1-(4x)^2}}\cdot(2)-0=\frac{2}{\sqrt{1-4x^2}}$

**8. Solution:**

$y=\left\{x+\sqrt{x^2+a^2}\right\}^n$ .....(1)

$\therefore \frac{dy}{dx}=n\left\{x+\sqrt{x^2+a^2}\right\}^{n-1}\cdot\frac{d}{dx}\left\{x+\sqrt{x^2+a^2}\right\}$

$=n\left\{x+\sqrt{x^2+a^2}\right\}^{n-1}\cdot\left[1+\frac{1}{2\sqrt{x^2+a^2}}(2x+0)\right]$

$=n\left\{x+\sqrt{x^2+a^2}\right\}^{n-1}\left\{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right\}$

$=\frac{n\left\{x+\sqrt{x^2+a^2}\right\}^n}{\sqrt{x^2+a^2}}=\frac{ny}{\sqrt{x^2+a^2}},$

[Using (1)]

which is true.

## Long Answer:

**1. Solution:**

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} a\sin\frac{\pi}{2}(x+1)$$

$$=\lim_{h\to 0} a\sin\frac{\pi}{2}(0-h+1)$$

$$=a\sin\frac{\pi}{2}(0-0+1)$$

$$=a\sin\frac{\pi}{2}=a.1=a$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\tan x-\sin x}{x^3}$$

$$=\lim_{h\to 0}\frac{\tan(0+h)-\sin(0+h)}{(0+h)^3}$$

$$=\lim_{h\to 0}\frac{\tan h-\sin h}{h^3}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}\frac{1-\cos h}{h^2}\cdot\frac{1}{\cos h}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{h^2}\cdot\lim_{h\to 0}\frac{1}{\cos h}$$

$$=1\cdot\frac{1}{2}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\cdot\frac{1}{\cos 0}$$

$$=1\cdot\frac{1}{2}(1)^2\cdot\frac{1}{1}=\frac{1}{2}\text{s}$$

Also f(0) = a sin π/2 (0+1)

= a sin π/2 = a(1) = a

For continuity,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

⇒ a = 1/2 = a

Hence, a = ½

**2. Solution:**

$$\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin^3 x}{3\cos^2 x}$$

$$=\lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}$$

$$=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$=\lim_{h\to 0}\frac{1+\cos^2 h+\cos h}{3(1+\cos h)}$$

$$=\frac{1+1+1}{3(1+1)}=\frac{1}{2}$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x)=\lim_{x\to\frac{\pi}{2}^+}\frac{q(1-\sin x)}{(\pi-2x)^2}$$

$$=\lim_{h\to 0}\frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}$$

$$=\lim_{h\to 0}\frac{q(1-\cos h)}{(\pi-\pi-2h)^2}$$

$$=\lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{q}{8}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

$$=\frac{q}{8}(1)^2=\frac{q}{8}$$

Also $f\left(\frac{\pi}{2}\right)=p$

For continuity $\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^+} f(x)$

$$=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow\frac{1^-}{2}=\frac{q}{8}=p$$

Hence p = 1/2 and q = 4

**3. Solution:**

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$$

$$=\lim_{x\to 0^-}\frac{\left(\sqrt{1+kx}-\sqrt{1-kx}\right)\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}$$

[Rationalising Numerator]

$$=\lim_{x\to 0^-}\frac{(1+kx)-(1-kx)}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}$$

$$= \lim_{x\to 0^-} \frac{2kx}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}$$

$$= \lim_{x\to 0^-} \frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}} \quad [\because x\neq 0]$$

$$= \frac{2k}{1+1} = k$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{2x+1}{x-1} = \lim_{h\to 0} \frac{2(0+h)+1}{(0+h)-1}$$

$$= \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$$

Also $f(0) = \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$

For continuity $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

⇒ k = -1 = -1

Hence k = -1

**4. Solution:**

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3ax + b)$

$= \lim_{h\to 0} (3a(1-h) + b]$

= 3a(1 – 0) + b

= 3a + b

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (5ax - 2b)$

$= \lim_{h\to 0} [5a(1+h) - 2b]$

= 5a (1+0) – 2b

= 5a – 2b

Also f(1) = 11

Since 'f' is continuous at x = 1,

∴ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

⇒ 3a + b = 5a – 2b = 11.

From first and third,

3a + b = 11 .............. (1)

From last two,

5a – 2b = 11 .............. (2)

Multiplying (1) by 2,

6a + 2b = 22 .............. (3)

Adding (2) and (3),

11a = 33

⇒ a = 3.

Putting in (1),

3(3) + b = 11

⇒ b = 11 – 9 = 2.

Hence, a = 3 and b = 2.

## Case Study Answers:

**1. Answer :**

i. (b) $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

**Solution:**

$$x^3 + x^2y + xy^2 + y^3 = 81$$

$$\Rightarrow 3^2 + x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (x^2+2xy+3y^2)\frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$$

ii. (c) $\frac{x-y}{x(1+\log x)}$

**Solution:**

$$x^y = e^{x-y} \Rightarrow y\log x = x - y$$

$$y\times\frac{1}{x} + \log x\cdot\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}[\log x + 1] = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x[1=\log x]}$$

iii. (d) $\frac{y}{x(y\cos y - 1)}$

Solution:

$$e^{\sin y} = xy \Rightarrow \sin y = \log x + \log y$$

⇒

$$\cos y\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[\cos y - \frac{1}{y}\right] = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x(y\cos y - 1)}$$

iv. (d) $\frac{\sin 2x}{\sin 2y}$

**Solution:**

$$\sin^2 x + \cos^2 y = 1$$

$$\Rightarrow 2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin 2x}{-\sin 2y} = \frac{\sin 2x}{\sin 2y}$$

v. (d) $\frac{y^2}{x(2-y\log x)}$

$y=\left(\sqrt{x}\right)^{\sqrt{x}^{\sqrt{x}\ldots\infty}} \Rightarrow y=\left(\sqrt{x}\right)^y$

$\Rightarrow \quad y=y\left(\log\sqrt{x}\right)\Rightarrow \log y=\frac{1}{2}(y\log x)$

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[y\times\frac{1}{x}+\log x\left(\frac{dy}{dx}\right)\right]$

$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{1}{y}-\frac{1}{2}\log x\right\}=\frac{1}{2}\frac{y}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{2x}\times\frac{2y}{(2-y\log x)}=\frac{y^2}{x(2-y\log x)}$

**2. Answer :**

i. (a) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

**Solution:**

Let $y=\cos\sqrt{x}$

$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\cos\sqrt{x}\right)=-\sin\sqrt{x}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$

$=-\sin\sqrt{x}\times\frac{1}{2\sqrt{x}}=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

ii. (a) $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

**Solution:**

Let $y=7^{x+\frac{1}{x}} \quad \therefore \frac{dy}{dx}=\frac{d}{dx}\left(7^{x+\frac{1}{x}}\right)$

$=7^{x+\frac{1}{x}}\cdot\log 7\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)=7^{x+\frac{1}{x}}\cdot\log 7\cdot\left(1-\frac{1}{x^2}\right)$

$=\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

iii. (a) $\frac{1}{2}\sec^2\frac{x}{2}$

**Solution:**

Let

$y=\sqrt{\frac{1-\cos x}{1+\cos x}}=\sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1+1}}=\tan\left(\frac{x}{2}\right)$

$\therefore \frac{dy}{dx}=\sec^2\frac{x}{2}\cdot\frac{1}{2}=\frac{1}{2}\sec^2\frac{x}{2}$

iv. (b) $\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$

**Solution:**

Let $y=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\therefore \frac{dy}{dx}=\frac{1}{b}\times\frac{1}{1+\frac{x^2}{b^2}}\times\frac{1}{b}+\frac{1}{a}\times\frac{1}{1+\frac{x^2}{a^2}}\times\frac{1}{a}$

$=\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$

v. (d) $\frac{2}{|x|\sqrt{x^2-1}}$

**Solution:**

Let $y=\sec^{-1}x+\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

Put $x=\sec\theta \Rightarrow \theta=\sec^{-1}x$

$\therefore y=\sec^{-1}(\sec\theta)+\operatorname{cosec}^{-1}\left(\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}\right)$

$=\theta+\sin^{-1}\left[\sqrt{1-\cos^2\theta}\right]$

$=\theta+\sin^{-1}(\sin\theta)=\theta+\theta=2\theta=2\sec^{-1}x$

$\therefore \frac{dy}{dx}=2\frac{d}{dx}(\sec^{-1}x)=2\times\frac{1}{|x|\sqrt{x^2-1}}$

$=\frac{2}{|x|\sqrt{x^2-1}}$

# Application of Derivatives | 6

**1.** If a quantity y varies with another quantity x, satisfying some rule y = f(x), then $\frac{dx}{dy}$ $(or\ f'(x)$ represents the rate of change of y with respect to x and $\frac{dy}{dx}]_{x=x_0}(or\ f'(x_0)$ represents the rate of change of y with respect to x at x = $x_0$.

**2.** If two variables x and y are varying with respect to another variable t, i.e., if x = f(t) and y =g(t) then by Chain Rule $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$, if $\frac{dx}{dt}\neq 0$.

A function f is said to be increasing on an interval (a, b) if $x_1 < x_2$ in (a, b) ⇒ $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$. Alternatively, if f'(x) > 0 for each x in, then f(x) is an increasing function on (a, b).

**3.** A function f is said to be increasing on an interval (a, b) if $x_1 < x_2$ in (a, b) ⇒ $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$. Alternatively, if f'(x) > 0 for each x in, then f(x) is an decreasing function on (a, b).

**4.** The equation of the tangent at $(x_0, y_0)$ to the curve y = f (x) is given by

$$y-y_0=\frac{dy}{dx}\Big]_{(x_0,y_0)}(x-x_0)$$

**5.** If $\frac{dy}{dx}$ does not exist at the point $(x_0, y_0)$, then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.

**6.** If tangent to a curve y = f(x) at $x = x_0$ is parallel to x-axis, then $\frac{dy}{dx}]_{x=x_0} = 0$

**7.** **Equation of the normal** to the curve y = f (x) at a point $(x_0, y_0)$, is given by

$$y-y_0=\frac{-1}{\frac{dy}{dx}\Big]_{(x_0,y_0)}}(x-x_0)$$

**8.** If $\frac{dy}{dx}$ at the point $(x_0, y_0)$, is zero, then equation of the normal is $x = x_0$.

**9.** If $\frac{dy}{dx}$ at the point $(x_0, y_0)$, does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$.

**10.** Let y = f (x), Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x, i.e., Δy = f (x + Δx) – f (x). Then $dy$ given by dy = f'(x) dx or $dy = \left(\frac{dy}{dx}\right) dx$ is a good of Δy when dx x = Δ is relatively small and we denote it by dy ≈ Δy.

**11.** A point c in the domain of a function f at which either f ′(c) = 0 or f is not differentiable is called a critical point of f.

**12.** **First Derivative Test:** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then,

i. If f ′(x) changes sign from positive to negative as x increases through c, i.e., if f ′(x) > 0 at every point sufficiently close to and to the left of c, and f ′(x) < 0 at every point sufficiently close to and to the right of c, then c is a point of local maxima.

ii. If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.

iii. If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflexion.

**13. Second Derivative Test:** Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

i. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The values f (c) is local maximum value of f.

ii. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, f (c) is local minimum value of f.

iii. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.

**14. Working rule for finding absolute maxima and/ or absolute minima**

**Step 1:** Find all critical points of *f* in the interval, *i.e.*, find points x where either $f'(x) = 0$ or f is not differentiable.

**Step 2:** Take the end points of the interval.

**Step 3:** At all these points (listed in Step 1 and 2), calculate the values of f.

**Step 4:** Identify the maximum and minimum values of *f* out of the values calculated in Step3.

This maximum value will be the absolute maximum value of *f* and the minimum value will be the absolute minimum value of *f*.

## Class : 12th Maths
## Chapter- 6 : Applications of Derivatives

# Applications of Derivatives

### Rate of Change Quantities

If a quantity if '$y$' varies with another quantity $x$ so that $y = f(x)$, then $\frac{dy}{dx}[f'(x)]$ represents the rate of change of $y$ w.r.t $x$ and $\frac{dy}{dx}\Big|_{x=x_0} (f'(x_0))$ represents the rate of change of $y$ w.r.t. $x$ at $x = x_0$

If '$x$' and '$y$' varies with another variable '$t$' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

For eg: if the radius of a circle, r = 5 cm, then the rate of change of the area of a circle per second w.r.t '$r$' is – $\frac{da}{dr}|_{r=5} = \frac{d}{dr}(\pi r^2)|_{r=5} = 2\pi r|_{r=5} = 10\pi$

### Approximations

Let $y=f(x)\Delta x$ be a small increment in '$x$' and $\Delta y$ be the small increment in $y$ corresponding to the increment in '$x$', i.e. $\Delta y = f(x+\Delta x) - f(x)$. Then, $\Delta y$ is given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$, is a good approximation of $\Delta y$ when $dx = \Delta x$ is relatively small and denote by $dy \approx \Delta y$. For eg: Let us approximate $\sqrt{36.6}$. To do this, we take: $y = \sqrt{x}, x = 36, \Delta x = 0.6$ then $\Delta y = \sqrt{x+dx} - \sqrt{x}$

$= \sqrt{36.6} - \sqrt{36}$

$= \sqrt{36.6} - 6 \Rightarrow \sqrt{36.6} = 6 + \bar{d}y$

Now, $dy$ is approximately $\Delta y$ and is given by $\bar{d}y$

$= \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05$. So, $\sqrt{36.6} \approx 6 + 0.05 = 6.05$.

### Increasing and decreasing functions

A function $f$ is said to be (i) increasing on $(a,b)$ if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \forall x_1, x_2 \in (a,b)$, and (ii) decreasing on $(a, b)$ if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2) \forall x_1 x_2 \in (a,b)$

If $f'(x) \geq 0 \forall x \in (a,b)$ then $f$ is increasing in $(a,b)$ and if $f'(x) \leq 0 \forall x \in (a,b)$, then $f$ is decreasing in $(a,b)$ For eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in R$. So, the function $f$ is strictly increasing on R.

### Maximum and Minima

A point C in the domain of '$f$' at which either $f'(c)=0$ or is not differentiable is called a critical point of $f$.

#### First Derivative test

Let $f$ be continuous at a critical point C in open I. Then **(i)** if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima. **(iii)** If $f'(x)$ does not change sign as '$x$' increases through C, then 'C' is called the point of inflection.

#### Second Derivative test

Let $f$ be a function defined on I and C∈I, $f$ is twice differentiable at C. Then **(i)** $x=C$ is a point of local max. If $f'(C)=0$ and $f''(C) < 0$, $f(C)$ is local max. of $f$. **(ii)** $x=C$ is a point of local min if $f'(C)=0$ and $f''(C) > 0$. $f(C)$ is local min of $f$. **(iii)** The test fails if $f'(C)=0$ and $f''(C)=0$

### Tangents and Normals

The equation of the tangent at $(x_0, y_0)$, to the curve $y = f(x)$ is given by $(y - y_0) = \frac{dy}{dx}\Big]_{(x_0, y_0)}(x - x_0)$ if $\frac{dy}{dx}$ does not exists at $(x_0, y_0)$, then the tangent at $(x_0, y_0)$ is parallel to the $y$–axis and its equation is $x = x_0$. If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to $x$–axis, then $\frac{dy}{dx}|_{x=x_0} = 0$.

### Equation of the normal to the curve

$y = f(x)$ at $(x_0, y_0)$ is $y - y_0 = -\frac{1}{\frac{dy}{dx}\Big|_{(x_0,y_0)}}(x - x_0)$ if $= \frac{dy}{dx}$ at $(x_0, y_0)$ is zero, then equation of the normal is $x = x_0$. If $\frac{dy}{dx}$ at $(x_0, y_0)$ does not exist, then the normal is parallel to $x$-axis and its equation is $y = y_0$ For eg: Let $y = x^3 - x$ be a curve, then the slope of the tangent to $y = x^3 - x$ at $x = 2$ is $\frac{dy}{dx}\Big|_{x=2} = 3x^2 - 1 = 3.2^2 - 1 = 11$

# Important Questions

## Multiple Choice questions-

1. The rate of change of the area of a circle with respect to its radius r at r = 6 cm is:

   (a) $10\pi$

   (b) $12\pi$

   (c) $8\pi$

   (d) $11\pi$

2. The total revenue received from the sale of x units of a product is given by $R(x)=3x^2+36x+5$. The marginal revenue, when x = 15 is:

   (a) 116

   (b) 96

   (c) 90

   (d) 126.

3. The interval in which $y = x^2 e^{-x}$ is increasing with respect to x is:

   (a) $(-\infty, \infty)$

   (b) (-2,0)

   (c) $(2, \infty)$

   (d) (0, 2).

4. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at x = 0 is

   (a) 3

   (b) $\frac{1}{3}$

   (c) -3

   (d) $-\frac{1}{3}$

5. The line y = x + 1 is a tangent to the curve $y^2 = 4x$ at the point:

   (a) (1, 2)

   (b) (2, 1)

   (c) (1, -2)

   (d) (-1, 2).

6. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of f(3.02) is:

   (a) 47.66

   (b) 57.66

   (c) 67.66

   (d) 77.66.

7. The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is:

   (a) $0.06\ x^3\ m^3$

   (b) $0.6\ x^3\ m^3$

   (c) $0.09\ x^3 m^3$

   (d) $0.9\ x^3\ m^3$

8. The point on the curve $x^2 = 2y$, which is nearest to the point (0, 5), is:

   (a) $(2\sqrt{2}, 4)$

   (b) $(2\sqrt{2}, 0)$

   (c) (0, 0)

   (d) (2, 2).

9. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

   (a) 0

   (b) 1

   (c) 3

   (d) $\frac{1}{3}$

10. The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is

    (a) $(\frac{1}{3})^{1/3}$

    (b) $\frac{1}{2}$

    (c) 1

    (d) 0

## Very Short Questions:

1. For the curve $y = 5x - 2x^3$, if increases at the rate of 2 units/sec., find the rate of change of the slope of the curve when x = 3.

2. Without using the derivative, show that the function f(x) = 7x – 3 is a strictly increasing function in R.

3. Show that function:

   $f(x) = 4x^3 - 18x^2 - 27x - 7$ is always increasing in R.

4. Find the slope of the tangent to the curve:

   $x = at^2$, $y = 2at$ t = 2.

5. Find the maximum and minimum values, if any, of the following functions without using derivatives:

(i) $f(x) = (2x-1)^2 + 3$

(ii) $f(x)= 16x^2 - 16x + 28$

(iii) $f(x) = -|x+ 1| + 3$

(iv) $f(x) = \sin 2x + 5$

(v) $f(x) = \sin (\sin x)$.

6. A particle moves along the curve $x^2 = 2y$. At what point, ordinate increases at die same rate as abscissa increases?

## Long Questions:

1. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall?
2. Find the angle of intersection of the curves $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2= 4$, at the point in the first quadrant.
3. Find the intervals in which the function: $f(x) = -2x^3 - 9x^2 - 12x + 1$ is (i) Strictly increasing (ii) Strictly decreasing.
4. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 meters. Find the dimensions of the window to admit maximum light through the whole opening.

## Assertion and Reason Questions:

**1.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

e) Both A and R are false.

**Assertion(A):** For each real 't', then exist a point C in $[t,t+\pi]$ such that $f'(C) = 0$

**Reason (R):** $f(t)=f(t+2\pi)$ for each real t

**2.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is not the correct explanation of A.

c) A is true but R is false.

d) A is false and R is true.

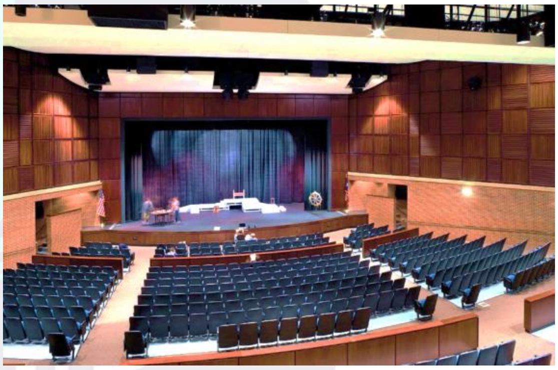
e) Both A and R are false.

**Assertion (A):** One root of $x^3-2x^2-1= 0$ and lies between 2 and 3.

**Reason(R):** If f(x) is continuous function and f [a], f[b] have opposite signs then at least one or odd number of roots of f(x)=0 lies between a and b.

## Case Study Questions:

**1.** An architecture design a auditorium for a school for its cultural activities. The floor of the auditorium is rectangular in shape and has a fixed perimeter P.



Based on the above information, answer the following questions.

i. If x any y represents the length and breadth of the rectangular region, then relation between the variable is

(a) $x + y = P$

(b) $x^2 + y^2 = P^2$

(c) $2(x+y) = P$

(d) $x + 2y = P$

ii. The area (A) of the rectangular region, as a function of x, can be expressed as.

(a) $A = px + \frac{x}{2}$

(b) $A = \frac{px + x^2}{2}$

(c) $A = \frac{px + 2x^2}{2}$

(d) $A = \frac{x^2}{2} + px^2$

iii. School's manager is interested in maximising the area of floor 'A' for this to be happen, the value of $x$ should be.

*(a)* $P$

(b) $\frac{P}{2}$

(c) $\frac{P}{3}$

(d) $\frac{P}{4}$

iv. The value of $y$, for which the area of floor is maximum, is

(a) $\frac{P}{2}$

(b) $\frac{P}{3}$

(c) $\frac{P}{4}$

(d) $\frac{P}{16}$

v. Maximum are of floor is

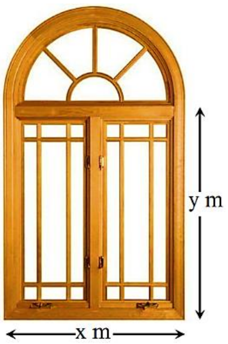
(a) $\frac{P^2}{16}$

(b) $\frac{P^2}{64}$

(c) $\frac{P^2}{4}$

(d) $\frac{P^2}{28}$

**2.** Rohan, a student of class XII, visited his uncle's flat with his father. He observes that the window of the house is in the form of a rectangle surmounted by a semicircular opening having perimeter 10m as shown in the figure.




Based on the above information, answer the following questions.

i. If $x$ and y represent the length and breadth of the rectangular region, then relation between $x$ and $y$ can be represented as.

(a) $x+y+\frac{\pi}{2}=10$

(b) $x+2y+\frac{\pi x}{2}=10$

(c) $2x + 2y = 10$

(d) $x+2y+\frac{\pi}{2}=10$

ii. The are (A) of the window can be given by

(a) $A=x-\frac{x^3}{8}-\frac{x^2}{2}$

(b) $A=5x-\frac{x^2}{8}-\frac{\pi x^2}{8}$

(c) $A=x+\frac{\pi x^3}{8}-\frac{3x^2}{8}$

(d) $A=5x+\frac{x^3}{2}+\frac{\pi x^2}{8}$

iii. Rohan is interested in maximizing the are of the whole window, for this to happen, the value of $x$ should be.

(a) $\frac{10}{2-\pi}$

(b) $\frac{20}{4-\pi}$

(c) $\frac{20}{4+\pi}$

(d) $\frac{10}{2+\pi}$

iv. Maximum area of the window is

(a) $\frac{30}{4+\pi}$

(b) $\frac{30}{4-\pi}$

(c) $\frac{50}{4-\pi}$

(d) $\frac{50}{4+\pi}$

v. For maximum value of A, the breadth of rectangular part of the window is

(a) $\frac{10}{4+\pi}$

(b) $\frac{10}{4-\pi}$

(c) $\frac{20}{4+\pi}$

(d) $\frac{20}{4-\pi}$

# Answer Key

## Multiple Choice questions-

1. **Answer:** (b) 12π
2. **Answer:** (d) 126.
3. **Answer:** (d) (0, 2).
4. **Answer:** (d) $-\frac{1}{3}$
5. **Answer:** (a) (1, 2)
6. **Answer:** (d) 77.66.
7. **Answer:** (c) 0.09 $x^3m^3$
8. **Answer:** (a) $(2\sqrt{2}, 4)$
9. **Answer:** (d) $\frac{1}{3}$
10. **Answer:** (c) 1

## Very Short Answer:

**1. Solution:**

The given curve is $y = 5x - 2x^3$

$\therefore \frac{dy}{dx} = 5 - 6x^2$

i.e., $m = 5 - 6x^2$,

where 'm' is the slope.

$\therefore \frac{dm}{dt} = -12x\frac{dx}{dt} = -12x\ (2) = -24x$

$\therefore \frac{dm}{dt}]_{x=3} = -24(3) = -72.$

Hence, the rate of the change of the slope = -72.

**2. Solution:**

Let $x_1$ and $x_2 \in$ R.

Now $x_1 > x_2$

$\Rightarrow 7x_1 > 7x_2$

$\Rightarrow 7x_1 - 3 > 7x_2 - 3$

$\Rightarrow f(x_1) > f(x_2)$.

Hence, 'f' is strictly increasing function in R.

**3. Solution:**

We have: $f(x) = 4x^3 - 18 \times 2 - 27x - 7$

$\therefore f(x) = 12x^2 - 36x + 27 = 12(x^2 - 3x) + 27$

$= 12(x^2 - 3x + 9/4) + 27 - 27$

$= 12(x - 3/2)^2 \forall\ x \in R.$

Hence, f(x) is always increasing in R.

**4. Solution:**

The given curve is $x - at^2$, $y = 2at$.

$\therefore \frac{dx}{dt} = 2at$

$\frac{dx}{dt} = 2a$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} =$

Hence, slope of the tangent at t = 2 is: $\frac{dy}{dx}]_{t=2} = \frac{1}{2}$

**5. Solution:**

(i) We have:

$f(x) = (2x - 1)^2 + 3.$

Here Df = R.

Now $f(x) \geq 3$.

$[\because (2x - 1)^2 \geq 0$ for all $x \in R]$

However, maximum value does not exist.

[∵ f(x) can be made as large as we please]

(ii) We have:

$f(x) = 16x^2 - 16x + 28.$

Here Df = R.

Now $f(x) = 16\ (x^2 - x + 14 + 24$

$= (16(x - \frac{1}{2})^2 + 24$

$\Rightarrow f(x) \geq 24.$

$[\because 16(x - 12)^2 \geq 0$ for all $x \in R$

Hence, the minimum value is 24.

However, maximum value does not exist.

[ ∵ f(x) can be made as large as we please]

(iii) We have:

$f(x) = -1x + 11 + 3$

$\Rightarrow f(x) \leq 3.$

$[\because -|x + 1| \leq 0]$

Hence, the maximum value = 3.

However, the minimum value does not exist.

[∵ f(x) can be made as small as we please]

(iv) We have :

$f(x) = \sin 2x + 5.$

Since $-1 \leq \sin 2x \leq 1$ for all $x \in R$,

$-1+5 \leq \sin 2x + 5 \leq 1+5$ for all $x \in R$

$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$ for all $x \in R$

$\Rightarrow 4 \leq f(x) \leq 6$ for all $x \in R$.

Hence, the maximum value = 6 and minimum value = 4.

(v) We have :

f(x) = sin (sin x).

We know that – 1 ≤ sin x ≤ 1 for all x ∈ R

⇒ sin(-1) ≤ sin(sinx) ≤ sin 1 for all x ∈ R

⇒ – sin 1 ≤ f(x) ≤ sin 1.

Hence, maximum value = sin 1 and minimum value = -sin 1.

**6. Solution:**

The given curve is $x^2 = 2y$ ...(1)

Diff.w.r.t.t, $2x\frac{dx}{dt} = 2\frac{dy}{dt}$

$\Rightarrow 2x\frac{dx}{dt} = 2\frac{dx}{dt}$

$\because \frac{dy}{dt} = \frac{dx}{dt}$ given

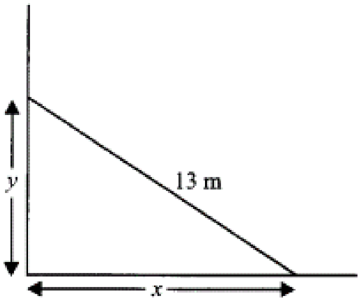
From (1), 1 = 2y ⇒ y = $\frac{1}{2}$

Hence, the reqd. point is $(1, \frac{1}{2})$

## Long Answer:

**1. Solution:**

Here, $\frac{dx}{dt} = 2\,cm/\text{sec}.$




Now, 169 = x2 + y2

$$\Rightarrow \quad y = \sqrt{169 - x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{169 - x^2}}(-2x)\frac{dx}{dt}$$

$$= -\frac{x}{\sqrt{169 - x^2}}(2)$$

$$\text{Hence, } \left.\frac{dy}{dt}\right]_{x=5} = \frac{-5}{\sqrt{169 - 25}}(2)$$

$$= \frac{-10}{12} = \frac{-5}{6}\text{ cm/sec.}$$

Hence, the height is decreasing at the rate of 5/6 cm/sec.

**2. Solution:**

The given curves are:

$x^2 + y^2 = 4$ ............(1)

$(x – 2)^2 + y^2 = 4$ ............. (2)

From (2),

$y = 4 – (x – 2)^2$

Putting in (1),

$x^2 + 4 - (x - 2)^2 = 4$

$\Rightarrow x^2 – (x – 2)^2 = 0$

⇒ (x + (x - 2)(x – x) + 2) = 0

⇒ (2x – 2)(2) = 0

⇒ x = 1.

Putting in (1),

$1 + y^2 = 4$

⇒ y = √3

∴ Point of intersection = (1, √3 )

Diff. (1) w.r.t. x, $2x + 2y\frac{dy}{dx} = 0$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{[1,\sqrt{3}]} = -\frac{1}{\sqrt{3}} = m_1$$

Diff. (2) w.r.t., $2(x-2) + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{[1,\sqrt{3}]} = \frac{1}{\sqrt{3}} = m_2$$

$$\text{So, } \tan\theta = \left|\frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)}\right| = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}} = \sqrt{3}.$$

Hence, $\theta = \frac{\pi}{3}$

**3. Solution:**

Given function is:

$f(x) = – 2x^3 – 9x^2 – 12x + 1.$

Diff. w.r.t. x,

$f'(x) = -6x^2 – 18x – 12$

= -6(x + 1) (x + 2).

Now, f'(x) – 0

⇒ x = -2, x = -1

⇒ Intervals are (-∞ – 2), (-2, -1) and (-1, ∞).

Getting f' (x) > 0 in (-2, -1)

and f'(x) < 0 in (-∞, -2) u (-1, ∞)

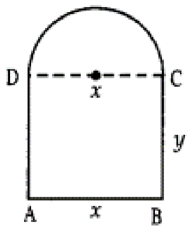
⇒ f(x) is strictly increasing in (-2, -1) and strictly decreasing in (-∞, 2) u (-1, ∞).

**4. Solution:**

Let 'x' and 'y' be the length and breadth of the rectangle ABCD.

Radius of the semi-circle = $\frac{x}{2}$.

Circumference of the semi-circle = $\frac{\pi x}{2}$




By the question, $x+2y+\frac{\pi x}{2}=10$

$\Rightarrow$ $2x + 4y + \pi x = 20$

$\Rightarrow$ $y=\frac{20-(2+\pi)x}{4}$ ....(i)

$\therefore$ Area of the figure $=xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$=x\frac{20-(2-\pi)x}{4}+\pi\frac{x^2}{8}$

[Using (1)]

Thus $A(x)=\frac{20x-(2+\pi)x^2}{4}+\frac{\pi x^2}{8}$

$\therefore$ $A'(x)=\frac{20-(2+\pi)(2x)}{4}+\frac{2\pi x}{8}$

and $A''(x)=\frac{-(2+\pi)2}{4}+\frac{2\pi}{8}$

$=\frac{-4-2\pi+\pi}{4}=\frac{-4-\pi}{4}$

or Max./Min. of A (x), A' (x) = 0

$\frac{20-(2+\pi)(2\pi)}{4}+\frac{2\pi x}{8}=0$

$20 - (2 + \pi)(2x) + \pi x = 0$

$20 + x(\pi - 4 - 2\pi) = 0$

$20 - x(4 + \pi) = 0$

$x=\frac{20}{4+\pi}$

and breadth = $y=\frac{20-(2+\pi)\frac{20}{4+\pi}}{4}$

$=\frac{80+20\pi-40-20\pi}{4(4+\pi)}=\frac{40}{4(4+\pi)}=\frac{10}{4+\pi}$

And radius of semi-circle $=\frac{10}{4+\pi}$

## Case Study Answers:

**1. Answer :**

i. (c) $2(x+y) = P$

**Solution:**

Perimeter of floor = 2(Length + breadth)

$\Rightarrow$ $P = 2(x+y)$

ii. (c) $A=\frac{px-2x^2}{2}$

**Solution:**

Area, A = length × breadth

$\Rightarrow$ $A = xy$

Since, $P = 2(x+y)$

$\Rightarrow$ $\frac{P-2x}{2}=y$

$\therefore$ $A=x\left(\frac{P-2x}{2}\right)$

$\Rightarrow$ $A=\frac{Px-2x^2}{2}$

iii. (d) $\frac{P}{4}$

**Solution:**

We have, $A=\frac{1}{2}(Px-2x^2)$

$\frac{dA}{dx}=\frac{1}{2}(P-4x)=0$

$\Rightarrow$ $P-4x=0 \Rightarrow x=\frac{P}{4}$

Clearly, at $x=\frac{P}{4}, \frac{d^2A}{dx^2}=-2<0$

$\therefore$ Area of maximum at $x=\frac{P}{4}$

iv. (c) $\frac{P}{4}$

**Solution:**

We have, $y=\frac{P-2x}{2}=\frac{P}{2}-\frac{P}{4}=\frac{P}{4}$

v. (a) $\frac{P^2}{16}$

**Solution:**

$A=xy=\frac{P}{4}\cdot\frac{P}{4}=\frac{P^2}{16}$

**2. Answer :**

i. (b) $x+2y+\frac{\pi x}{2}=10$

**Solution:**

Given, perimeter of window = 10m

$\therefore$ $x + y + y$ + perimeter of semicircle = 10

$\Rightarrow$ $x+2y+\pi\frac{2}{2}=10$

ii. (b) $A=5x-\frac{x^2}{8}-\frac{\pi x^2}{8}$

**Solution:**

$$A=x\cdot y+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$=x\left(5-\frac{x}{2}-\frac{\pi x}{4}\right)+\frac{1}{2}\frac{\pi x^2}{4}$$

[$\because$ From (i), $y=5-\frac{x}{2}-\frac{\pi x}{4}$]

$$=5x-\frac{x^2}{2}-\frac{\pi x^2}{4}+\frac{\pi x^2}{8}=5x-\frac{x^2}{2}-\frac{\pi x^2}{8}$$

iii. (c) $\frac{20}{4+\pi}$

**Solution:**

We have, $A=5x-\frac{x^2}{2}-\frac{\pi r^2}{8}$

$\Rightarrow \quad \frac{dA}{dx}=5-x-\frac{\pi x}{4}$

Now, $\Rightarrow \frac{dA}{dx}=0$

$\Rightarrow \quad 5=x+\frac{\pi x}{4}$

$\Rightarrow \quad x(4+\pi)=20$

$\Rightarrow \quad x=\frac{20}{4+\pi}$

$\left[\text{Clearly, } \frac{d^2A}{dx^2}<0 \text{ at } x=\frac{20}{4+\pi}\right]$

iv. (d) $\frac{50}{4+\pi}$

**Solution:**

At $x=\frac{20}{x}=\frac{20}{4+\pi}$

$$A=5\left(\frac{20}{4+\pi}\right)-\left(\frac{20}{4+\pi}\right)^2\frac{1}{2}-\frac{\pi}{8}\left(\frac{20}{4+\pi}\right)^2$$

$$=\frac{100}{4+\pi}-\frac{200}{(4+\pi)^2}-\frac{50\pi}{(4+\pi)^2}$$

$$\frac{(4+\pi)(100)-200-50\pi}{(4+\pi)^2}=\frac{400+100\pi-200-50\pi}{(4+\pi)^2}$$

$$\frac{200+50\pi}{(4+\pi)}=\frac{50(4+\pi)}{(4+\pi)}=\frac{50}{4+\pi}$$

v. (a) $\frac{10}{4+\pi}$

**Solution:**

We have, $y=5-\frac{x}{2}-\frac{\pi x}{4}=5-x\left(\frac{1}{2}+\frac{\pi}{4}\right)$

$$=5-x\left(\frac{2+\pi}{4}\right)=5-\left(\frac{20}{4+\pi}\right)\left(\frac{2+\pi}{4}\right)$$

$$=5-5\frac{(2+\pi)}{4+\pi}=\frac{20+5\pi-10-5\pi}{4+\pi}=\frac{10}{4+\pi}$$

## Assertion and Reason Answers:

**1.** (a) Both A and R are true and R is the correct explanation of A.

**Solution:**

Given that $f(x)=2+\cos x$

Clearly $f(x)$ is continuous and differentiable everywhere Also $f'(x)=-\sin x \Rightarrow f'(x=0)$

$\Rightarrow$ $-\sin x=0 \Rightarrow x=n\pi$

$\therefore$ These exists $C \in [t, t+\pi]$ for $t \in R$

such that $f'(C)=0$

$\therefore$ Statement-1 is true Also

f(x) being periodic function of period $2\pi$

$\therefore$ Statement-2 is true, but Statement-2 is not a correct explanation of Statement -1.

**2.** (a) Both A and R are true and R is the correct explanation of A.

**Solution:**

Given $f(x)=x^3-2x^2-1=0$

Here, $f(2)=(2)^3-2(2)^2-1=8-8-1=-1$

and $f(3)=(3)^3-2(3)^2-1=27-18-1=8$

$\therefore$ $f(2)f(3)=(-1)8=-8<0$

$\Rightarrow$ One root of f(x) lies between 2 and 3

$\therefore$ Given Assertion is true Also Reason R is true and valid reason

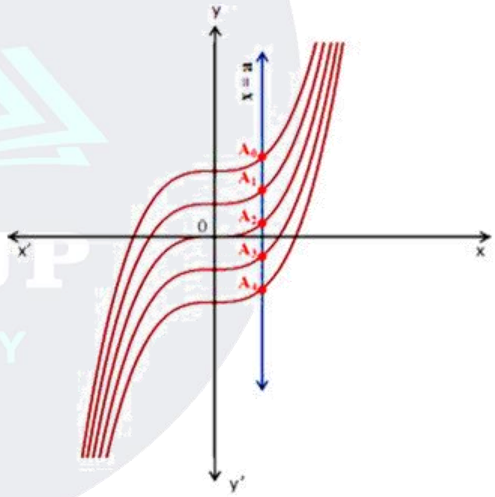
$\therefore$ Both A and R are correct and R is correct explanation of A.

# Integrals 7

1. Integration is the inverse process of differentiation. The process of finding the function from its primitive is known as integration or antidifferentiation.
2. The problem of finding a function whenever its derivative is given leads to indefinite form of integrals.
3. The problem of finding the area bounded by the graph of a function under certain conditions leads to a definite form of integrals.
4. Indefinite and definite integrals together constitute **Integral Calculus**.
5. Indefinite integral $\int f(x)dx = F(x) + C$, where F(x) is the antiderivative of f(x).
6. Functions with same derivatives differ by a constant.
7. $\int f(x)dx$ means integral of f with respect to x, f(x) is the integrand, x is the variable of integration and C is the constant of integration.
8. Geometrically indefinite integral is the collection of family of curves, each of which can be obtained by translating one of the curves parallel to itself.

   Family of curves representing the integral of $3x^2$

   $\int f(x)dx = F(x) + C$, represents a family of curves where different values of C correspond to different members of the family, and these members are obtained by shifting any one of the curves parallel to itself.




9. **Properties of antiderivatives**

   $\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$

   $\int kf(x)dx-k\int f(x)dx$ for any real number k

   $\int[k_1f_1(x)+k_2f_2(x)+...+k_nf_n(x)]dx=k_1\int f_1(x)dx+k_2\int f_2(x)dx+...+k_n\int f_n(x)dx$

   where $k_1, k_2, ...k_n$ are real numbers and $f_1, f_2, ...f_n$ are real functions.
10. Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.
11. **Comparison between differentiation and integration**
    1. Both are operations on functions.
    2. Both satisfy the property of linearity.
    3. All functions are not differentiable and all functions are not integrable.
    4. The derivative of a function is a unique function, but the integral of a function is not.
    5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.
    6. The derivative is defined at a point P and the integral of a function is defined over an interval.

7. Geometrical meaning: The derivative of a function represents the slope of the tangent to the corresponding curve at a point. The indefinite integral of a function represents a family of curves placed parallel to each other having parallel tangents at the points of intersection of the family with the lines perpendicular to the axis.
8. The derivative is used for finding some physical quantities such as the velocity of a moving particle when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time I is known.
9. Differentiation and integration, both are processes involving limits.
10. By knowing one antiderivative of function f, an infinite number of antiderivatives can be obtained.

**12.** Integration can be done by using many methods. Prominent among them are

1. Integration by substitution
2. Integration using partial fractions
3. Integration by parts
4. Integration using trigonometric identities.

**13.** A change in the variable of integration often reduces an integral to one of the fundamental integrals. Some standard substitutions are

$x^2+a^2$; substitute x = a tan θ

$\sqrt{x^2-a^2}$ ; substitute x = a sec θ

$\sqrt{a^2-x^2}$ ; substitute x = a sin θ or a cos θ

**14.** A function of the form $\frac{P(x)}{Q(x)}$ is known as a rational function. Rational functions can be integrated using partial fractions.

**15.** **Partial fraction decomposition or partial fraction expansion** is used to reduce the degree of either the numerator or the denominator of a rational function.

**16.** **Integration using partial fractions**

A rational function $\frac{P(x)}{Q(x)}$ can be expressed as the sum of partial fractions if $\frac{P(x)}{Q(x)}$. This takes any of the forms:

- $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}, a\neq b$
- $\frac{px+q}{(x-a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$
- $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
- $\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
- $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ cannot be factorized further.

**17.** To find the integral of the product of two functions, integration by parts is used. I and II functions are chosen using the ILATE rule:

I - inverse trigonometric

L - logarithmic

A - algebraic

T - trigonometric

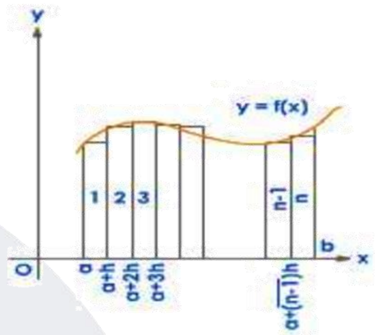
E - exponential is used to identify the first function.

**18. Integration by parts**

Integral of the product of two functions = (first function) × (integral of the second function) – integral of [(differential coefficient of the first function) × (integral of the second function)].

$\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x).\int f_2(x)dx\right]dx$, where $f_1$ and $f_2$ are functions of x.

**19.** Definite integral $\int_a^b f(x)dx$ of the function f(x) from limits a to b represents the area enclosed by the graph of the function f(x), the x-axis and the vertical markers x = 'a' and x = 'b'.




**20. Definite integral as the limit of a sum:** The process of evaluating a definite integral by using the definition is called integration as the limit of a sum or integration from first principles.

**21.** Method of evaluating $\int_a^b f(x)dx$

i. Calculate antiderivative F(x)

ii. Calculate F(b ) - F(a)

**22.** Area function

A (x) = $\int_a^x f(x)dx$ , if x is a point in [a, b].

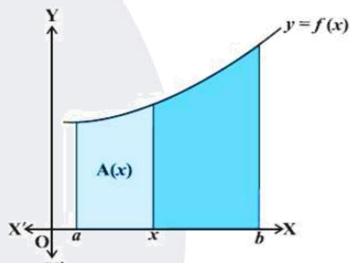



Fig 7.3

**23.** Fundamental Theorem of Integral Calculus

- **First fundamental theorem** of integral calculus: If area function, $A(x) = \int_a^x f(x)dx$ for all x ≥ a, and f is continuous on [a, b]. Then A'(x) = f(x) for all x ∈ [a, b]
- **Second fundamental theorem** of integral calculus: Let f be a continuous function of x in the closed interval [a, b] and let F be antiderivative of $\frac{d}{dx}f(x) = f(x)$ for all x in domain of *f*, then

$$\int_a^b f(x)dx = [f(x)+c]_a^b = F(b)-F(a)$$

## Top Formulae

**1. Some Standard Integrals**

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$
- $\int dx = x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sin x\, dx = -\cos x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int cosec^2 x\, dx = -\cot x + C$
- $\int cosec\, x \cot x\, dx = -cosec\, x + C$

- $\int \cos x\,dx = \sin x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
- $\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log a} + C$
- $\int \frac{1}{x} dx = \log|x| + C$
- $\int \tan x dx = \log|\sec x| + C$
- $\int \cot x dx = \log|\sin x| + C$
- $\int \sec x dx = \log|\sec x + \tan x| + C$
- $\int \text{cosec}\, x dx = \log|\text{cosec}\, x - \cot x| + C$
- $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$
- $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + C$
- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
- $\int a^{bx+c} dx = \frac{1}{b}.\frac{a^{bx+c}}{\log a} + C, a > 0, a \neq 1$
- $\int \sin(ax+b) dx = -\frac{1}{a}\cos(ax+b) + C$
- $\int \cos(ax+b) dx = \frac{1}{a}\sin(ax+b) + C$
- $\int \tan(ax+b) dx = \frac{1}{a}\log|\sec(ax+b)| + C$
- $\int \cot(ax+b) dx = \frac{1}{a}\log|\sin(ax+b)| + C$
- $\int \sec^2(ax+b) dx = \frac{1}{a}\tan(ax+b) + C$

- $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$
- $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$
- $\int \text{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\text{cosec}(ax+b)+C$
- $\int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)|+C$
- $\int \text{cosec}(ax+b)dx = \frac{1}{a}\log|\text{cosec}(ax+b)-\cot(ax+b)|+C$

**2. Integral of some special functions**

- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$
- $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$
- $\int -\frac{dx}{x^2+a^2} = \frac{1}{a}\cot^{-1}\frac{x}{a}+C$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$
- $\int -\frac{dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{x}{a}+C$
- $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$
- $\int -\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{cosec}^{-1}\frac{x}{a}+C$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C$
- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+C$
- $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+C$
- $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}+C$
- $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\log\left|x+\sqrt{a^2+x^2}\right|+C$
- $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|+C$

**3. Integration by parts**

**(i)** $\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x).\int f_2(x)dx\right]dx$, where $f_1$ and $f_2$ are functions of x.

**(ii)** $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+C$

**4. Integral as the limit of sums:**

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right] \text{ where } h=\frac{b-a}{n}.$$

**5. Different methods of integration**

i. Evaluation of integrals of the form $\frac{p(x)}{(ax+b)^n}$, n ∈ N, where p(x) is a polynomial step:

(a) Check whether degree of p(x) ≥ or ≤ n

(b) If degree of p(x) < n, then express p(x) in the form $A_0 + A_1(ax+b)+A_2(ax+b)^2+...+A_{n-1}(ax+b)^{n-1}$

(c) Write $\frac{p(x)}{(ax+b)^n}$ as $\frac{A_0}{(ax+b)^n}+\frac{A_1}{(ax+b)^{n-1}}+\frac{A_2}{(ax+b)^{n-2}}+...+\frac{A_{n-1}}{(ax+b)}$

(d) Evaluate

$$\int\frac{p(x)}{(ax+b)^n}dx = A_0\int\frac{1}{(ax+b)^n}dx + A_1\int\frac{1}{(ax+b)^{n-1}}+A_2\int\frac{1}{(ax+b)^{n-2}}dx+...+A_{n-1}\int\frac{1}{(ax+b)}dx$$

(e) If degree of p(x) > n, then divide p(x) by $(ax+b)^n$ and express $\frac{p(x)}{(ax+b)^n}$ as $q(x)+\frac{r(x)}{(ax+b)^n}$, where degree of r(x) is less than n

(f) Use steps (b) and (c) to evaluate $\int\frac{r(x)}{(ax+b)^n}dx$

ii. Evaluation of integrals of the form $\int(ax+b)\sqrt{cx+d}dx$

Step:

(a) Represent (ax + b) in terms of (cx + d) as follows:

(ax + b) = A (cx + d) + B

(b) Find A and B by equating coefficients of like powers of x on both sides

(c) Replace (ax + b) by A (cx + d) + B in the given integral to obtain

$$\int(ax+b)\sqrt{cx+d}\,dx = \int\left|A(cx+d)+B\right|\sqrt{cx+d}dx$$

$$= A\int(cx+d)^{\frac{3}{2}}dx + B\int\sqrt{cx+d}\,dx$$

$$= \frac{2A}{5c}(cx+d)^{\frac{5}{2}}+\frac{2B}{5c}(cx+d)^{\frac{3}{2}}+C$$

**6.** Evaluation of integrals of the form $\int\frac{(ax+b)}{\sqrt{cx+d}}dx$

Step:

i. Represent (ax + b) in terms of (cx + d) as follows:

(ax + b) = A (cx + d) + B

ii. Find A and B by equating coefficients of like powers of x on both sides

iii. Replace (ax + b) by A(cx + d) + B in the given integral to obtain

$$\int \frac{(ax+b)}{\sqrt{cx+d}}dx = \int \frac{|A(cx+d)+B|}{\sqrt{cx+d}}dx$$

$$A\int \sqrt{cx+d}\,dx + B\int \frac{1}{\sqrt{cx+d}}dx$$

$$\frac{2A}{3c}(cx+d)^{\frac{3}{2}} + \frac{2B}{c}(cx+d)^{\frac{1}{2}} + C$$

**7.** Evaluation of integrals of the form $\int xdx$ , $\int cos^m xdx, where\ m \ \leq \ 4$

Let us express $\sin^m x$ and $\cos^m x$ in terms of sines and cosines of multiples of x by using the following identities:

(a) $\sin^2 x = \frac{1-\cos 2x}{2}$

(b) $\cos^2 x = \frac{1+\cos 2x}{2}$

(c) $\sin 3x = 3 \sin x - 4 \sin^3 x$

(d) $\cos 3x = 4 \cos^3 x - 3 \cos x$

**8.** Evaluation of integrals of the form

$$\int sinmx.cosnxdx, \int sin\ mx\ .sinnxdx, \int cos\ mx.cosnxdx$$

Let us use the following identities:

(a) 2 sin A cos B = sin (A + B) + sin (A – B)

(b) 2 cos A cos B = cos (A + B) + cos (A – B)

(c) 2 cos A sin B = sin (A + B) – sin (A – B)

(d) 2 sin A sin B = cos (A – B) – cos (A + B)

**9.** Evaluation of integrals of the form $\int \frac{f'(x)}{f(x)}dx$

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

**10.** Evaluation of integrals of the form

$\int (ax+b)^n p(x)dx, \int \frac{p(x)}{(ax+b)^n}dx$, where p(x) is a polynomial and n is a positive rational number

Steps:

i. Substitute ax + b = v or $x = \frac{v-b}{a}$ and $dx = \frac{1}{a}dv$

ii. Now integrate with respect to v by using $\int v^n dv = \frac{v^{n+1}}{n+1} + C$

iii. Replace v by ax + b

**11.** Evaluation of integrals of the form $\int \tan^m x \sec^{2n} x dx, \int \cot^m x.\text{cosec}^{2n} x dx$, m, n ∈ N

Steps:

i. Rewrite the given integral as $I = \int \tan^m x.(\sec^2 x)^{(n-1)} \sec^2 x dx$

ii. Substitute tan x = v and $\sec^2 x\, dx = dv$

Therefore,

$$I=\int \tan^m x.\left(\sec^2 x\right)^{(n-1)}\sec^2 x\,dx$$

$$=\int \tan^m x.\left(1+\tan^2 x\right)^{(n-1)}\sec^2 x\,dx$$

$$=\int v^m.\left(1+v^2\right)^{(n-1)}dv$$

iii. Use the binomial theorem to expand $\left(1+v^2\right)^{(n-1)}$ in step (ii) and integrate

iv. Replace v by tan x in step (iii)

**12.** Evaluation of integrals of the form $\int \tan^{2m+1} x \sec^{2n+1} x\,dx, \int \cot^m x.\text{cosec}^{2n} x dx$, where m and n are non-negative integers

Steps:

i. Rewrite the given integral as $I=\int \left(\tan^2 x\right)^m.(\sec x)^{2n}\sec x.\tan x\,dx$

ii. Put sec x = v and sec x tan x dx = dv

Therefore,

$$I=\int \left(\sec^2 x-1\right)^m.(\sec x)^{2n}\sec x.\tan x\,dx$$

$$=\int \left(v^2-1\right)^m v^n dv$$

iii. Use the binomial theorem to expand $\left(v^2-1\right)^m$ in step (ii) and integrate

iv. Replace v by sec x in step (iii)

**13.** Evaluation of integrals of the form $\int x\,.\,cos^n\,xdx, where\ m,n \in N$

Steps:

i. Check the exponents of sinx and cosx

ii. If the exponent of sinx is an odd positive integer, then put cosx = v

If the exponent of cosx is an odd positive integer, then put sinx = v

If the exponents of both sinx and cosx are odd positive integers, then put either sinx = v or cosx = v

If the exponents of both sinx and cosx are even positive integers, then rewrite $\sin^m x \cos^n x$ in terms of sines and cosines of multiples of x by using trigonometric results.

iii. Evaluate the integral in step (ii)

**14.** Evaluation of integrals of the form $\int \sin^m x.\cos^n xdx$, where m, n ∈ Q, such that m + n is negative even integer.

Steps:

i. Represent the integrand in terms of tan x and $\sec^2$ x by dividing the numerator and denominator by $\cos^k$x, where k = – (m + n)

ii. Put tan x = v

**15.** Evaluation of integrals of the form $\int \frac{dx}{ax^2+bx+c}$

Steps:

i. Multiply and divide the integrand by $x^2$ and make the coefficient of $x^2$ unity

ii. Observe the coefficient of x

iii. Add and subtract $\left(\frac{1}{2}coefficient\ x\right)^2$ to the expression in the denominator

iv. Express the expression in the denominator in the form $\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}$

v. Use the appropriate formula to integrate.

**16.** Evaluation of integrals of the form $\int \frac{dx}{ax^2+bx+c}$

Steps:

i. Multiply and divide the integrand by $x^2$ and make the coefficient of $x^2$ unity

ii. Observe the coefficient of x

iii. Add and subtract $\left(\frac{1}{2}coefficient\ x\right)^2$ inside the square root

iv. Express the expression inside the square root in the form $\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}$

v. Use the appropriate formula to integrate.

**17.** Evaluation of integrals of the form $\int \frac{px+q}{ax^2+bx+c}dx$

Steps:

i. Rewrite the numerator as follows:

$$px+q=A\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+B$$

$\Rightarrow$ px + q = A {2ax + b} + B

ii. Find the values of A and B by equating the coefficients of like powers of x on both sides

iii. Substitute px + q by A {2ax + b} + B in the given integral

Therefore, $\int\frac{px+q}{ax^2+bx+c}dx=A\int\frac{2ax+b}{ax^2+bx+c}dx+B\int\frac{1}{ax^2+bx+c}dx$

iv. Integrate the right-hand side and substitute the values of A and B

**18.** Evaluation of integrals of the form $\int \frac{px+q}{ax^2+bx+c}dx$ where p(x) is a polynomial degree greater than or equal to 2

Steps:

i. Divide the numerator by the denominator, and rewrite the integrand as $q(x)+\frac{r(x)}{ax^2+bx+c}$, where r(x) is a linear function of x

ii. Thus, $\int\frac{px+q}{ax^2+bx+c}dx=\int q(x)dx+\int\frac{r(x)}{ax^2+bx+c}dx$

iii. Integrate the second integral on the right-hand side and apply the appropriate method

**19.** Evaluation of integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Steps:

i. Rewrite the numerator as follows:

$$px+q=A\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+B$$

$\Rightarrow$ px + q = A {2ax + b} + B

ii. Find the values of A and B by equating the coefficients of like powers of x on both sides

iii. Substitute px + q by A {2ax + b} + B in the given integral

Therefore, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

iv. Integrate the right-hand side and substitute the values of A and B

**20.** Evaluation of integrals of the form

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{dx}{a+b\sin^2 x + c\cos^2 x}$$

Steps:

i. Divide the numerator and denominator by $\cos^2 x$

ii. In the denominator, replace $\sec^2 x$ by $1 + \tan^2 x$

iii. Substitute $\tan x = v$; $\sec^2 x dx = dv$

iv. Apply the appropriate method to integrate the integral $\int \frac{dv}{av^2+bv+c}$

**21.** Evaluation of integrals of the form

$$\int \frac{dx}{a\sin x + b\cos x}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x + c\cos x}$$

Steps:

i. Substitute $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

ii. In the numerator, replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$

iii. Substitute $\tan\frac{x}{2} = v;\ \frac{1}{2}\sec^2\frac{x}{2}dx = dv$

iv. Apply the appropriate method to integrate the integral $\int \frac{dv}{av^2+bv+c}$

**22.** Alternate method: Evaluation of integrals of the form $\int \frac{dx}{a\sin x + b\cos x}$

Steps:

Substitute a = r cos θ, b = r sin θ; where $r = \sqrt{a^2+b^2}, = \theta\tan^{-1}\left(\frac{b}{a}\right)$

⇒ a sin x + b cos x = r cos θ sin x + r sin θ cos x = r sin (x + θ)

$$\therefore \int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + c$$

**23.** Evaluation of integrals of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$

Steps:

i. Substitute

Numerator = A (Differentiation of denominator) + B (Denominator) That is a sin x + b cos x = A (c cos x – d sin x) + B(c sin x + d cos x)

ii. Compare the coefficients of sin x and cos x on both the sides and get the values of A and B

iii. Hence, the value of the integral $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$ is

$$= A\log|c\sin x + d\cos x| + \mu x + C$$

**24.** Evaluation of integrals of the form $\int \frac{a\sin x + b\cos x + c}{m\sin x + n\cos x + p}dx$

Steps:

i. Substitute: Numerator = A (Differentiation of denominator) + B (Denominator) + K

That is a sin x + b cos x + c = A (m cos x – n sin x) + B(m sin x + n cos x + p) + K

ii. Compare the coefficients of sin x and cos x and constant terms on both the sides and get the values of A, B and K

iii. Replace the integrand by A (m cos x – n sin x) + B(m sin x + n cos x + p) + K

iv. Hence, the value of the integral $\int \frac{a\sin x + b\cos x + c}{m\sin x + n\cos x + p}dx$ is

$$= A\log|m\sin x + n\cos x + p| + Bx + p\int \frac{1}{m\sin x + n\cos x + p}dx$$

v. Evaluate the integral on the right-hand side by any appropriate method

**25.** Evaluation of integrals of the form $\int e^x\left[f(x) + f'(x)\right]dx$

Steps:

i. Write the given integral as $\int e^x\left[f(x) + f'(x)\right]dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$

ii. Find the integration for the first term by parts

iii. Cancel out the second integral with the second term obtained by integration by parts

iv. Thus, the above result holds true for $e^{kx}$

$$\int e^{kx}\left[f(x) + f'(x)\right]dx = e^{kx} + C$$

**26.** Evaluation of the integrals of the form $\int \sqrt{ax^2 + bx + cdx}$

Steps:

i. Take 'a' common inside the square root so as to get $x^2 + \frac{b}{a}x + \frac{c}{a}$

ii. Add and subtract the appropriate term to $x^2 + \frac{b}{a}x + \frac{c}{a}$ to get the term $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$

iii. Now evaluate the integral using the appropriate formulae

**27.** Evaluation of the integrals of the form $\int (mx + n)\sqrt{ax^2 + bx + cdx}$

Steps:

i. Rewrite mx + n as mx + n = $A\frac{d}{dx}\left(ax^2 + bx + c\right) + B$

That is, mx + n = A(2ax + b) + B

ii. Equate the coefficients of x and constant terms on both sides to get the values of A and B

iii. Substitute mx + n by A(2ax + b) + B

iv. Now evaluate the integral using the appropriate formulae

**28.** Evaluation of the integrals of the form $\int \frac{x^2+1}{x^4+kx^3+1}dx, \int \frac{x^2-1}{x^4+kx^2+1}dx, \int \frac{1}{x^4+kx^2+1}dx$ where k ∈ R

Steps:

i. Divide the numerator and the denominator by $x^2$

ii. Write the denominator of the integrand in the form of $\left(x+\frac{1}{x}\right)^2 \pm m^2$

iii. Write $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in the numerator

iv. Put $x+\frac{1}{x}=v \; or \; x-\frac{1}{x}=v$

v. Evaluate the integral using the appropriate formula

**29.** Evaluation of integration of irrational algebraic functions, $\int \frac{f(x)}{(gx+h)\sqrt{mx+n}}dx$ where g, h, m, n ∈ R

Step: Put $mx+n=v^2$ to evaluate the integral

**30.** Evaluation of integration of irrational algebraic functions, $\int \frac{f(x)}{(rx^2+gx+h)\sqrt{mx+n}}dx$ where r, g, h, m, n ∈ R

Step: Put $mx+n=v^2$ to evaluate the integral

**31.** Evaluation of integration of irrational algebraic functions, $\int \frac{1}{(gx+h)\sqrt{mx^2+nx+p}}dx$ where g, h, m, n, p ∈ R

i. Put $gx+h=\frac{1}{v}$ to evaluate the integral

**32.** Evaluation of integration of irrational algebraic functions, $\int \frac{1}{(gx^2+h)\sqrt{mx^2+n}}dx$ where g, h, m, n ∈ R

step:

i. Put $x=\frac{1}{v}$ Therefore, $I=\int \frac{-vdv}{(g+hv^2)\sqrt{m+nv^2}}$

Now substitute $m + nv^2 = w^2$

**33. Properties of definite integrals**

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$ in particular, $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
- $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x),$

  $= 0 \qquad \text{if } f(2a-x) = -f(x)$
- $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x) = f(x),$

  $= 0 \qquad \text{if } f(-x) = -f(x)$

**Class : 12th Maths**
**Chapter- 7 : Integrals**

# Integrals

## Integration

It is the inverse of differentiation. Let, $\frac{d}{dx}F(x)=f(x)$. Then $\int f(x)dx=F(x)+c$, 'c' constant of integral. These integrals are called indefinite or general integrals.

Properties of indefinite integrals are

(i) $\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$, (ii) $\int kf(x)dx=k\int f(x)dx$,

For eg : $\int(3x^2+2x)dx=x^3+x^2+c$ where $k$ is real.

## Integration by substitution

The method in which we change the variable to some other variable is called the method of substitution

$\int \tan x dx=\log|\sec x|+c$  $\int \cot x dx=\log|\sin x|+c$

$\int \sec x dx=\log|\sec x+\tan x|+c$  $\int \operatorname{cosec} x dx=\log|\operatorname{cosec} x-\cot x|+c.$

## Integration of some special functions

(i) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ (ii) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(iii) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (iv) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

(v) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$ (vi) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+c.$

## Some Standard integrals

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq-1$ like, $\int dx=x+c$

(ii) $\int\cos x dx=\sin x+c$ (iii) $\int\sin x dx=-\cos x+c$

(iv) $\int\sec^2 x dx=\tan x+c$ (v) $\int\operatorname{cosec}^2 x dx=-\cot x+c$

(vi) $\int\sec x\tan x dx=\sec x+c$ (vii) $\int\operatorname{cosec}x\cot x dx=-\operatorname{cosec}x+c$

(viii) $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$ (ix) $\int\frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+c$

(x) $\int\frac{dx}{1+x^2}=\tan^{-1}x+c$ (xi) $\int\frac{dx}{1+x^2}=-\cot^{-1}x+c$

(xii) $\int e^x dx=e^x+c$ (xiii) $\int a^x dx=\frac{a^x}{\log a}+c$

(xiv) $\int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+c$ (xv) $\int\frac{dx}{x\sqrt{x^2-1}}=-\operatorname{cosec}^{-1}x+c$

(xvi) $\int\frac{1}{x}dx=\log|x|+c$

## Some special type of integrals

(i) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c.$

(ii) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$

(iii) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$

## First fundamental theorem of integral calculus

Let the area function be defined by $A(x)=\int_a^x f(x)dx\ \forall x\geq a$, where $f$ is continuous on $[a,b]$ then $A'(x)=f(x)\ \forall x\in[a,b]$.

## Integration by parts

$$\int f_1(x)f_2(x)dx=f_1(x)\int f_2(x)dx-\int\left[\frac{d}{dx}f_1(x)\int f_2(x)dx\right]dx$$

## Second fundamental theorem of integral Calculus

Let f be a continuous function of $x$ defined on $[a,b]$ and let $F$ be another function such that $\frac{d}{dx}F(x)=f(x)\ \forall x\in$ domain of $f$, then $\int_a^b f(x)dx=[F(x)+c]_a^b=F(b)-F(a)$. This is called the definite integral of $f$ over the range $[a,b]$, where $a$ and $b$ are called the limits of integration, $a$ being the lower limit and $b$ be the upper limit.

## Integration by partial fractions

A rational function of the form $\frac{P(x)}{Q(x)}(Q(x)\neq0)=T(x)+\frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}, a\neq b.$

(ii) $\frac{px+q}{(x+a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

## Example

$$\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx$$

$$=2\int_0^{\pi/4}\sin^2 x\,dx$$

$$=2\int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=\int_0^{\pi/4}(1-\cos 2x)dx$$

$$=\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$=\frac{\pi}{4}-\frac{1}{2}$$

# Important Questions

## Multiple Choice Questions-

1. The anti-derivative of $(\sqrt{x} + \frac{1}{\sqrt{x}})$ equals

(a) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + c$

(b) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^{2} + c$

(c) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$

(d) $\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + c$

2. If $\frac{1}{dx}$ (f(x)) = $4x^3 - \frac{3}{x^4}$ such that f(2) = 0 then f(x) is ..............

(a) $x^4 + \frac{1}{x^3} - \frac{129}{8}$

(b) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(c) $x^4 + \frac{1}{x^3} + \frac{129}{8}$

(d) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

3. $\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

(a) $10^x - x^{10} + c$

(b) $10^x + x^{10} + c$

(c) $(10^x - x^{10})^{-1} + c$

(d) $\log(10^x + x^{10}) + c$.

4. $\int \frac{10x^9 + 10^z \log_e 10}{x^{10} + 10^x} dx$ equals

(a) tan x + cot x + c

(b) tan x - cot x + c

(c) tan x cot x + c

(d) tan x - cot 2x + c.

5. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

(a) tan x + cot x + c

(b) tan x + cosec x + c

(c) -tan x + cot x + c

(d) tan x + sec x + c.

6. $\int \frac{e^x(1+x)}{\cos^2(xe^2)} dx$ is equal to

(a) $-\cot(xe^x) + c$

(b) $\tan(xe^x) + c$

(c) $\tan(e^x) + c$

(d) $\cot(e^x) + c$

7. $\int \frac{dx}{x^2 + 2x + 2}$ equals

(a) $x \tan^{-1}(x + 1) + c$

(b) $\tan^{-1}(x + 1) + c$

(c) $(x + 1)\tan^{-1} x + c$

(d) $\tan^{-1} x + c$.

8. $\int \frac{dx}{\sqrt{9 - 25x^2}}$ equals

(a) $\sin^{-1}\left(\frac{5x}{3}\right) + c$

(b) $\frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + c$

(c) $\frac{1}{6}\log\left(\frac{3+5x}{3-5x}\right) + c$

(d) $\frac{1}{30}\log\left(\frac{3+5x}{3-5x}\right) + c$

9. $\int \frac{x\,dx}{(x-1)(x-2)}$ equals

(a) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + c$

(b) $\log\left|\frac{(x-2)^2}{x-1}\right| + c$

(c) $\log\left|\left(\frac{x-2}{x-1}\right)^2\right| + c$

(d) $\log|(x-1)(x-2)| + c$

10. $\int \frac{dx}{x(x^2+1)}$ equals

(a) $\log|x| - \frac{1}{2}\log|(x^2+1)| + c$

(b) $\frac{1}{2}\log|x| + \frac{1}{2}\log|(x^2+1)| + c$

(c) $-\log|x|+\frac{1}{2}\log\left|\left(x^2+1\right)\right|+c$

(d) $\log|x|+\log\left|\left(x^2+1\right)\right|+c$

## Very Short Questions:

1. Find $\int \frac{3+3\cos x}{x+\sin x} dx$
2. Find: $\int (\cos^2 2x - \sin^2 2x)$ dx.
3. Find: $\int \frac{dx}{\sqrt{5-4x-2x^2}}$
4. Evaluate $\int \frac{x^3-1}{x^2}$ dx
5. Find: $\int \frac{\sin^2 x-\cos^2 x}{\sin x \cos x}$ dx
6. Write the value of $\int \frac{dx}{x^2+16}$
7. Evaluate: $\int (x^3 + 1)$ dx.
8. Evaluate: $\int_0^{\pi/2} e^x(\sin x - \cos x)$ dx.
9. Evaluate: $\int_0^2 \sqrt{4-x^2}\, dx$
10. Evaluate: If f(x) = $\int_0^x t \sin t\, dt$, then write the value of f' (x).

## Short Questions:

1. Evaluate: $\int \frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$
2. Find: $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$ dx
3. Find: $\int \sqrt{1-\sin 2x}dx, \frac{\pi}{4}<x<\frac{\pi}{2}$
4. Find $\int \sin x . \log \cos x$ dx
5. Find: $\int \frac{(x^2+\sin^2 x)\sec^2 x}{1+x^2}$ dx
6. Evaluate $\int \frac{e^x(x-3)}{(x-1)^3}$ dx
7. Find $\int \sin^{-1}(2x)$ dx
8. Evaluate: $\int_{-\pi}^{\pi}(1-x^2) \sin x \cos 2x$ dx.

## Long Questions:

1. Evaluate: $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x \cos^2 x}$ dx
2. Integrate the function $\frac{\cos(x+a)}{\sin(x+b)}$ w.r.t. x.
3. Evaluate: $\int x^2 \tan^{-1} x$ dx.
4. Find: $\int [\log(\log x) + \frac{1}{(\log x)^2}]$ dx

## Case Study Questions-

1. Integration is the process of finding the antiderivative of a function. In this process, we are provided with the derivative of a function and asked to find out the function (i.e., Primitive) Integration is the inverse process of differentiation.

   Let f (x) be a function of x. If there is a function g(x), such that d/dx (g(x)) = f (x), then g(x) is called an integral of f (x) w.r.t x and is denoted by $\int$f (x)dx =g(x)+c, where c is constant of integration.

   **(i)** $\int(3x+4)^3$ dx is equal to:

   (a) $\frac{(3x+4)^4}{12}+c$

   (b) $\frac{3(3x+4)^4}{4}+c$

   (c) $\frac{3(3x+4)^2}{2}+c$

   (d) $\frac{3(3x+4)^2}{4}+c$

   **(ii)** $\int\frac{(x+1)^2}{x(x^2+1)}dx$ is equal to

   (a) $\log|x|+c$

   (b) $\log|x|+2\tan^{-1}x+c$

   (c) $-\log|x^2+1|+c$

   (d) $\log\left|x\left(x^2+1\right)\right|+c$

   **(iii)** $\int\sin^2(x)$ dx is equal to:

   (a) $\frac{x}{2}+\frac{\sin 2x}{4}+c$

   (b) $\frac{x}{2}-\frac{\sin 2x}{4}+c$

   (c) $x+\frac{\sin 2x}{2}+c$

   (d) $x-\frac{\sin 2x}{2}+c$

   **(iv)** $\int\tan^2(x)$ dx is equal to:

   (a) tan x + x + c

   (b) – tan x – x + c

   (c) – tan x + x + c

   (d) tan x – x + c

**(v)** $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to

(a) 2 tan 2x + c

(b) –2 tan 2x + c

(c) – 2 cot 2x + c

(d) 2 cot 2x + c

**2.** When the integrated can be expressed as a product of two functions, one of which can be differentiated and the other can be integrated, then we apply integration by parts. If f(x) = first function (that can be differentiated) and g(x) = second function (that can be integrated), then the preference of this order can be decided by the word "ILATE", where

I stand for Inverse Trigonometric Function

L stands for Logarithmic Function

A stands for Algebraic Function

T stands for Trigonometric Function

E stands for Exponential Function, then

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left\{\frac{d}{dx}f(x)\int g(x)dx\right\}dx$$

**(i)** $\int x.\sin 3x\, dx =$

(a) $\frac{x\cos 3x}{3} - \frac{\sin 3x}{a} + c$

(b) $-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + c$

(c) $\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + c$

(d) $-\frac{x\cos 3x}{3} - \frac{\sin 3x}{9} + c$

**(ii)** $\int \log(x + 1)\, dx =$

(a) $\log(x+1) - x + c$

(b) $x\log(x+1) - x + c$

(c) $x\log(x+1) - \log(x+1) + x + c$

(d) $x\log(x+1) + \log(x+1) - x + c$

**(iii)** $\int x^2 e^{3x} dx =$

(a) $\frac{e^{3x}}{9}(9x^2 + 6x + 2) + c$

(b) $\frac{e^{3x}}{9}(9x^2 - 6x + 2) + c$

(c) $\frac{e^{3x}}{27}(9x^2 + 6x + 2) + c$

(d) $\frac{e^{3x}}{27}(9x^2 - 6x + 2) + c$

**(iv)** $\int (f(x)g''(x) - f''(x)g(x))\, dx =$

(a) $f(x)g'(x) - f'(x)g(x) + c$

(b) $f(x)g'(x) + f'(x)g(x) + c$

(c) $f'(x)g(x) - f(x)g'(x) + c$

(d) $\frac{f(x)}{g'(x)} + c$

# Answer Key

## Multiple Choice questions-

**1.** **Answer:** (c) $\frac{2}{3}\times\frac{2}{3} + 2\times\frac{1}{2} + C$

**2.** **Answer:** (a) $x^4 + \frac{1}{x^3} - \frac{129}{8}$

**3.** **Answer:** (d) log $(10^x + x^{10})$ + c.

**4.** **Answer:** (b) tan x – cot x + c

**5.** **Answer:** (a) tan x + cot x + c

**6.** **Answer:** (b) tan $(xe^x)$ + c

**7.** **Answer:** (b) $\tan^{-1}(x + 1)$ + c

**8.** **Answer:** (b) $\frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + c$

**9.** **Answer:** (b) $\log\left|\frac{(x-2)^2}{x-1}\right| + c$

**10.** **Answer:** (a) $\log|x| - \frac{1}{2}\log(x^2 + 1) + c$

## Very Short Answer:

**1.** **Solution:**

I = $\int \frac{3+3\cos x}{x+\sin x} dx$ = 3 log |x + sin x| + c.

[$\because$ Num. = $\frac{d}{dx}$ denom.]

**2.** **Solution:**

I = $\int \cos 4x\, dx = \frac{\sin 4x}{4}$ + c

**3. Solution:**

$$I=\int\frac{dx}{\sqrt{5-4x-2x^2}}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2-(x+1)^2}}$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left[\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right]+c.$$

**4. Solution:**

$$\int\frac{x^3-1}{x^2}dx=\int\left(x-\frac{1}{x^2}\right)dx$$

$$=\int xdx-\int x^{-2}dx$$

$$=\frac{x^2}{2}-\frac{x^{-1}}{-1}+c$$

$$=\frac{x^2}{2}+\frac{1}{x}+c,\ x\neq 0.$$

**5. Solution:**

$$I=\int\frac{\sin^2 x-\cos^2 x}{\sin x\cos x}dx$$

$$=\int\frac{\sin^2 x}{\sin x\cos x}dx-\int\frac{\cos^2 x}{\sin x\cos x}dx$$

$$=\int\tan x\,dx-\int\cot x\,dx$$

$$=\log|\sec x|-\log|\sin x|+c.$$

**6. Solution:**

$$\int\frac{dx}{x^2+16}=\int\frac{dx}{4^2+x^2}$$

$$|\text{"From}:\int\frac{dx}{a^2+x^2}\text{"}$$

$$=\frac{1}{4}\tan^{-1}\frac{x}{4}+c.$$

**7. Solution:**

$I=\int_{-2}^{2}x^3\,dx+\int_{-2}^{2}1\cdot dx=I_1$

$\Rightarrow 0+\left[x^2\right]_{-2}^{2}$

[$\because I_1$ is an odd function] = 2 – (–2) = 4.

$\Rightarrow$ 2 – (-2) = 4.

**8. Solution:**

$\int_0^{\pi/2}e^x(\sin x-\cos x)dx$

$\int_0^{\pi/2}e^x(-\cos x+\sin x)dx$

I" Form: $\int e^x(f(x)+f'(x)\,dx)$"

$$=\left[e^x(-\cos x)\right]_0^{\pi/2}$$

$=-e^{\pi/2}\cos\frac{\pi}{2}+e^0\cos 0$

$=-e^{\pi/2}(0)+(1)(1)$

= -0 + 1 = 1

**9. Solution:**

$$\int_0^2\sqrt{4-x^2}dx=\int_0^2\sqrt{2^2-x^2}dx$$

$$|\text{"From}:\int\sqrt{a^2-x^2}dx\text{"}$$

$$=\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$=[0+2\sin^{-1}(1)]-[0+0]$

$=2\sin^{-1}(1)=2(\pi/2)=\pi$

**10. Solution:**

We have: $f(x)=\int_0^x t\sin t\,dt$.

$f'(x)=x\sin x.\frac{d}{dx}(x)-0$

[Property XII; Leibnitz's Rule]

= x sin x. (1)

= x sin x.

## Short Answer:

**1. Solution:**

$$I=\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$$

$$=\int\frac{(1-2\sin^2 x)+2\sin^2 x}{\cos^2 x}dx$$

$$=\int\frac{1}{\cos^2 x}dx=\int\sec^2 x\,dx$$

$$=\tan x+c.$$

**2. Solution:**

$I=\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Put tan x = t so that $\sec^2 x\,dx=dt$.

$\therefore I=\int\frac{dt}{\sqrt{t^2+2^2}}$

$=\log|t+\sqrt{t^2+4}|+C$

$=\log|\tan x+\sqrt{t^2+4}|+C$

**3. Solution:**

$$I=\int\sqrt{1-\sin 2x dx}$$

$$=\int\sqrt{\left(\sin^2 x+\cos^2 x\right)-2\sin x\cos x\,dx}$$

$$=\int\sqrt{\left(\sin x-\cos x\right)^2}\,dx$$

$$=\int\left(\sin x-\cos x\right)dx$$

$$=-\cos x-\sin x+c.$$

**4. Solution:**

∫sinx . log cos x dx

Put cox x = t

so that – sin x dx = dt

i.e., sin x dx = – dt.

∴ I = -∫log t.1dt

= -[ log t.t – ∫ 1/t. t dt ]

[Integrating by parts]

= – [t log t – t] + C = f(1 – log t) + C

= cos x (1 – log (cos x)) + C.

**5. Solution:**

$$I=\int\frac{\left(x^2+\sin^2 x\right)\sec^2 x}{1+x^2}dx$$

$$=\int\left\{\frac{\left(1+x^2\right)+\left(\sin^2 x-1\right)}{1+x^2}\right\}\sec^2 x\,dx$$

$$=\int\left(1-\frac{\cos^2 x}{1+x^2}\right)\sec^2 dx$$

$$=\int\sec^2 dx-\int\frac{1}{1+x^2}dx$$

$$=\tan x-\tan^{-1}x+c.$$

**6. Solution:**

$$I=\int\frac{e^x\left(x-3\right)}{\left(x-1\right)^3}dx$$

$$=\int e^x\left[\frac{\left(x-1\right)-2}{\left(x-1\right)^3}\right]dx$$

$$=\int e^x\left[\frac{1}{\left(x-1\right)^2}-\frac{2}{\left(x-1\right)^3}\right]$$

$$\left|\text{"From}:\int e^x\left[f\left(x\right)+f'\left(x\right)\right]dx\text{"}\right.$$

$$=\frac{e^x}{\left(x-1\right)^2}+c.$$

**7. Solution:**

$$I=\int\sin^{-1}\left(2x\right)dx=\int\sin^{-1}\left(2x\right).1dx$$

$$=\sin^{-1}\left(2x\right).x-\int\frac{1}{\sqrt{1-4x^2}}\left(2\right)\left(x\right)dx$$

[Integrating by Parts]

$$=x\sin^{-1}\left(2x\right)-\int\frac{2x}{\sqrt{1-4x^2}}dx$$

$$=x\sin^{-1}\left(2x\right)+\frac{1}{4}\int\left(1-4x^2\right)^{-1/2}\left(-8x\right)dx$$

$$=x\sin^{-1}\left(2x\right)+\frac{1}{4}\frac{\left(1-4x^2\right)^{1/2}}{1/2}+c$$

$$=x\sin^{-1}\left(2x\right)+\frac{1}{2}\sqrt{1-4x^2}+c.$$

**8. Solution:**

Here, f(x) = (1 - $x^2$) sin x $\cos^2$ x.

f(x) = (1 – $x^2$) sin (-x) $\cos^2$ (-x)

= – (1 – $x^2$) sin x $\cos^2$ x

= -f(x)

⇒ f is an odd function.

Hence, I = 0.

## Long Answer:

**1. Solution:**

$$\int\frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$$

$$\left(\sin^2 x+\cos^2 x\right)^3-$$

$$=\int\frac{3\sin^2 x\cos^2 x\left(\sin^2 x+\cos^2 x\right)}{\sin^2 x\cos^2 x}dx$$

$$\left[\because a^3+b^3=\left(a+b\right)^3-3ab\left(a+b\right)\right]$$

$$=\int\frac{\left(1\right)^3-3\sin^2 x\cos^2 x\left(1\right)}{\sin^2 x\cos^2 x}dx$$

$$=\int\left(\frac{1}{\sin^2 x\cos^2 x}-3\right)dx$$

$$=\int\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-3\right)dx$$

*(Note this step)*

$$=\int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}-3\right)dx$$

$$=\int\sec^2 dx+\int cosec^2 dx-3\int 1dx$$

$$=\tan x-\cot x-3x+c.$$

**2. Solution:**

$$I=\int\frac{\cos(x+a)}{\sin(x+b)}dx$$

$$=\int\frac{\cos\left(\overline{x+b}+\overline{a-b}\right)}{\sin(x+b)}dx$$

$$\cos(x+b)\cos(a-b)$$

$$=\int\frac{-\sin(x+b)\sin(a-b)}{\sin(x+b)}dx$$

$$=\cos(a-b)\int\frac{\cos(x+a)}{\sin(x+b)}dx-\sin(a-b)\int 1.dx$$

$$=\cos(a-b)\log\left|\sin(x+b)\right|-\sin(a-b).x+c.$$

**3. Solution:**

$$\int x^2\tan^{-1}x\,dx=\int\tan^{-1}x.\,x^2\,dx$$

$$=\tan^{-1}x.\frac{x^3}{3}-\int\frac{1}{1+x^2}.\frac{x^3}{3}\,dx$$

*[Integrating by parts]*

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\frac{x^3}{1+x^2}\,dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int x\,dx+\frac{1}{6}\int\frac{2x}{1+x^2}\,dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}.\frac{x^2}{2}+\frac{1}{6}\log\left|1+x^2\right|+c$$

$$\left[\because\frac{d}{dx}\left(1+x^2\right)=2x\right]$$

$$=\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log\left(1+x^2\right)+c.$$

$$\left[\because x^2\geq 0\Rightarrow 1+x^2>0\;\therefore\left|1+x^2\right|=1+x^2\right]$$

**4. Solution:**

Let $\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx$

$=\int\log(\log x)\,dx+\int\frac{1}{(\log x)^2}\,dx$ ...... (1)

Let $I = I_1 + I_2$

Now $I_1=\int\log(\log x)\,dx$

$=\int\log(\log x)\,1\,dx$

$=\log(\log x).x-\int\frac{1}{\log x\cdot x}x.dx$

(Integrating by parts)

$=x\log(\log x)-\int\frac{1}{\log x}dx$ .......... (2)

Let $I_1 = I_3 + I_4$

And $I_4=\int\frac{1}{\log x}\,dx$

$$=\int\frac{1}{\log x}.1dx$$

$$=\frac{1}{\log x}.x-\int-\left(\frac{1}{\log x}\right)^2.\frac{1}{x}.x\,dx$$

(Integrating by parts)

$$=\frac{x}{\log x}+\int\frac{1}{(\log x)^2}\,dx$$

Putting in (2),

$$I_1=x\log(x)-\frac{x}{\log x}-\int\frac{1}{(\log x)^2}dx$$

Putting in (1),

I = x log (log x)

$$-\frac{x}{\log x}-\int\frac{1}{(\log x)^2}dx+\int\frac{1}{(\log x)^2}dx$$

$$=x\log(\log x)-\frac{x}{\log x}+c$$

$$=x\left(\log(\log x)-\frac{1}{\log x}\right)+c.$$

## Case Study Answers-

**1.**

(i) (a) $\frac{(3x+4)^4}{12}+c$

(ii) (b) $\log|x|+2\tan^{-1}x+c$

(iii) (b) $\frac{x}{2}-\frac{\sin 2x}{4}+c$

(iv) (d) tan x − x + c

(v) (c) $-2\cot 2x+c$

**2.**

(i) (b) $-\frac{x\cos 3x}{3}+\frac{\sin 3x}{9}+c$

(ii) (d) $x\log(x+1)+\log(x+1)-x+c$

(iii) (d) $\frac{e^{3x}}{27}\left(9x^2-6x+2\right)+c$

(iv) (a) $f(x)g'(x)-f'(x)g(x)+c$

# Application of Integrals 8

1. **Elementary area:** The area is called elementary area which is located at any arbitrary position within the region which is specified by some value of x between a and b.

2. The area of the region bounded by the curve y = f (x), x-axis and the lines x = a and x = b (b > a) is given by the formula: $Area = \int_z^b vdx = \int_z^b f(x)dx$

3. The area of the region bounded by the curve $x = \theta(y)$ , y-axis and the lines y = c, y = d is given by the formula: $Area = \int_c^b xdy = \int_c^b \theta(y)dy$

4. The area of the region enclosed between two curves y = f (x), y = g (x) and the lines x = a, x = b is given by the formula, Area = $\int_a^b [f(x) - g(x)]dx$, where $f(x) \geq g(x)$ in [a, b].

5. If $f(x) \geq g(x)$ in [a, c] and $f(x) \leq g(x)$ in [c, b], a < c < b, then we write the areas as:

$$Area = \int_a^b [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

**Class : 12th Maths**
**Chapter- 8 : Applications of Integrals**

## Applications of Integrals

### Area between two curves

The area of the region enclosed between two curves $y = f(x), y = g(x)$ and the lines $x=a$, $x=b$ is given by

$$A = \int_a^b [f(x) - g(x)]dx, \text{ where } f(x) \geq g(x) \text{ in } [a,b]$$

For eg: To find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$

(0,0) and (1,1) are points of intersection of $y = x^2$ and $y^2 - x$ and, $y^2 = x \Rightarrow y = \sqrt{x} = f(x)$, and $y = x^2 = g(x)$, where $f(x) \geq g(x)$ in $[0, 1]$.

$$\text{Area, } A = \int_0^1 [f(x) - g(x)]dx$$

$$= \int_0^1 [\sqrt{x} - x^2]dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ Sq. units.}$$

if $f(x) \geq g(x)$ in $[a,c]$ and $f(x) \leq g(x)$ in $[c, \ b]$, $a < c < b$, then the area is

$$A = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

### Area under simple curves

The area of the region bounded by the curve $x = f(y)$, $y-axis$ and the lines $y=c$ and $y = d \ (d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(y)dy$.

For $eg$ : the area bounded by $x = y^3$, $y-axis$ in the I quadrant and the lines $y=1$ and $y=2$ is

$$A = \int_1^2 x\,dy = \int_1^2 y^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4} \text{ Sq. units}$$

The area of the region bounded by the curve $y = f(x)$, $x$- $axis$ and the lines $x=a$ and $x = b \ (b > a)$ is given by

$$A = \int_a^b y\,dx \text{ or } \int_a^b f(x)dx.$$

For $eg$ : the area bounded by $y=x^2$, $x$- $axis$ in I quadrant and the lines $x=2$ and $x=3$ is -

$$A = \int_2^3 y\,dx = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27 - 8) = \frac{19}{3} \text{ Sq.units.}$$

# Important Questions

## Multiple Choice Questions-

1. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines x = 0 and x = 2 is
   (a) π
   (b) $\frac{\pi}{2}$
   (c) $\frac{\pi}{3}$
   (d) $\frac{\pi}{4}$

2. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line y = 3 is
   (a) 2
   (b) $\frac{9}{4}$
   (c) $\frac{9}{3}$
   (d) $\frac{9}{2}$

3. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line x + y = 2 is
   (a) 2 (π – 2)
   (b) π – 2
   (c) 2π – 1
   (d) 2 (π + 2).

4. Area lying between the curves $y^2 = 4x$ and y = 2 is:
   (a) $\frac{2}{3}$
   (b) $\frac{1}{3}$
   (c) $\frac{1}{4}$
   (d) $\frac{3}{4}$

5. Area bounded by the curve $y = x^3$, the x-axis and the ordinates x = -2 and x = 1 is
   (a) -9
   (b) $-\frac{15}{4}$
   (c) $\frac{15}{4}$
   (d) $\frac{17}{4}$

6. The area bounded by the curve y = x|x|, x-axis and the ordinates x = -1 and x = 1 is given by
   (a) 0
   (b) $-\frac{1}{3}$
   (c) $\frac{2}{3}$
   (d) $\frac{4}{3}$

7. The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is
   (a) $\frac{4}{3}(4\pi - \sqrt{3})$
   (b) $\frac{1}{3}(4\pi + \sqrt{3})$
   (c) $\frac{2}{3}(8\pi - \sqrt{3})$
   (d) $\frac{4}{3}(8\pi + \sqrt{3})$

8. The area enclosed by the circle $x^2 + y^2 = 2$ is equal to
   (a) 4π sq. units
   (b) $2\sqrt{2}\,\pi$ sq. units
   (c) $4\pi^2$ sq. units
   (d) 2π sq. units.

9. The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to
   (a) $\pi^2ab$
   (b) πab
   (c) $\pi a^2b$
   (d) $\pi ab^2$.

10. The area of the region bounded by the curve $y = x^2$ and the line y = 16 is
    (a) $\frac{32}{3}$
    (b) $\frac{256}{3}$
    (c) $\frac{64}{3}$
    (d) $\frac{128}{3}$

## Very Short Questions:

1. Find the area of region bounded by the curve $y = x^2$ and the line y = 4.
2. Find the area bounded by the curve $y = x^3$, x = 0 and the ordinates x = -2 and x = 1.

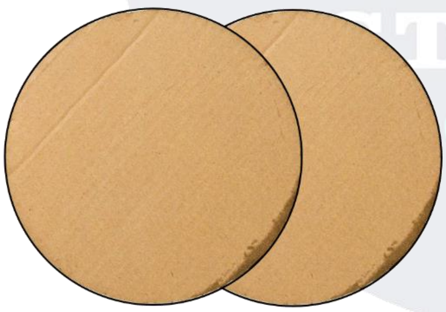
3. Find the area bounded between parabolas $y^2 = 4x$ and $x^2 = 4y$.
4. Find the area enclosed between the curve $y = \cos x$, $0 \le x \le \frac{\pi}{4}$ and the co-ordinate axes.
5. Find the area between the x-axis curve $y = \cos x$ when $0 \le x < 2$.
6. Find the ratio of the areas between the center $y = \cos x$ and $y = \cos 2x$ and x-axis for $x = 0$ to $x = \frac{\pi}{3}$
7. Find the areas of the region:
   $\{(x,y): x^2 + y^2 \le 1 \le x + 4\}$

## Long Questions:

1. Find the area enclosed by the circle: $x^2 + y^2 = a^2$.
2. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.
3. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = Y$ and Y-axis.
4. Find the area of region: $\{(x,y): x^2 + y^2 < 8, x^2 < 2y\}$.

## Case Study Questions:

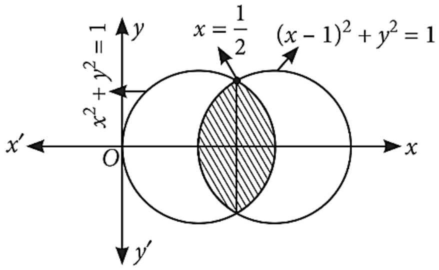
**1.** Ajay cut two circular pieces of cardboard and placed one upon other as shown in figure. One of the circle represents the equation $(x - 1)^2 + y^2 = 1$, while other circle represents the equation $x^2 + y^2 = 1$.

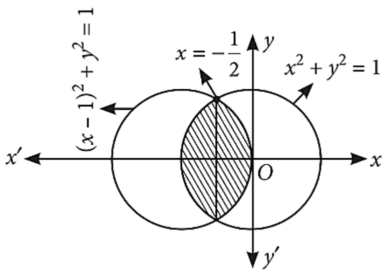


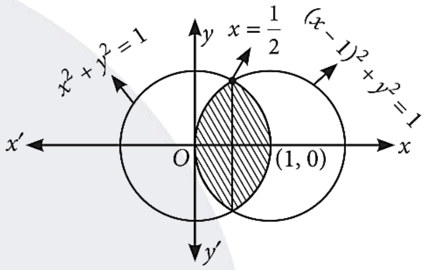
Based on the above information, answer the following questions.

i. Both the circular pieces of cardboard meet each other at
   (a) $x = 1$
   (b) $x = \frac{1}{2}$
   (c) $x = \frac{1}{3}$
   (d) $x = \frac{1}{4}$

ii. Graph of given two curves can be drawn as.

(a)



(b)



(c)



(d) None of these

iii. Value of $\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\,dx$ is
   (a) $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$
   (b) $\frac{\pi}{6} + \frac{\sqrt{3}}{8}$
   (c) $\frac{\pi}{2} + \frac{\sqrt{3}}{4}$
   (d) $\frac{\pi}{2} - \frac{\sqrt{3}}{4}$

iv. Value of $\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx$ is.
   (a) $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$
   (b) $\frac{\pi}{6} + \frac{\sqrt{3}}{8}$
   (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$
   (d) $\frac{\pi}{2} - \frac{\sqrt{3}}{4}$

v. Area of hidden portion of lower circle is.

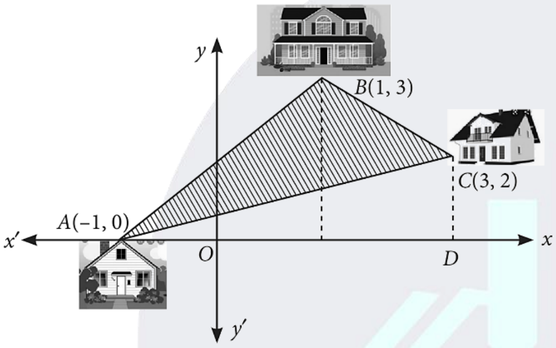
(a) $\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\right)$ sq. units

(b) $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{8}\right)$ sq. units

(c) $\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right)$ sq. units

(d) $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units

**2.** Location of three houses of a society is represented by the points A(-1, 0), B(1, 3) and C(3, 2) as shown in figure.




Based on the above information, answer the following questions

(i) Equation of line AB is.

(a) $y=\frac{3}{2}(x+1)$

(b) $y=\frac{3}{2}(x-1)$

(c) $y=\frac{1}{2}(x+1)$

(d) $y=\frac{1}{2}(x-1)$

(ii) Equation of line BC is.

(a) $y=\frac{1}{2}x-\frac{7}{2}$

(b) $y=\frac{3}{2}x-\frac{7}{2}$

(c) $y=\frac{-1}{2}x+\frac{7}{2}$

(d) $y=\frac{3}{2}x+\frac{7}{2}$

(iii) Area of region ABCD is.

(a) 2 sq. units

(b) 4 sq. units

(c) 6 sq. units

(d) 8 sq. units

(iv) Area of $\triangle ADC$ is,

(a) 4 sq. units

(b) 8 sq. units

(c) 16 sq. units

(d) 32 sq. units

(v) Area of $\triangle ABC$ is.

(a) 3 sq. units

(b) 4 sq. units

(c) 5 sq. units

(d) 6 sq. units

# Answer Key

## Multiple Choice Questions-

1. Answer: (a) π
2. Answer: (a) 2
3. Answer: (b) π – 2
4. Answer: (b) $\frac{1}{3}$
5. Answer: (b) - $\frac{15}{4}$
6. Answer: (c) $\frac{2}{3}$
7. Answer: (c) $\frac{2}{3}(8\pi-\sqrt{3})$
8. Answer: (d) 2π sq. units.
9. Answer: (b) πab
10. Answer: (b) $\frac{256}{3}$

## Very Short Answer:

1. **Solution:** $\frac{32}{2}$ sq. units.
2. **Solution:** $\frac{17}{4}$ sq. units.
3. **Solution:** $\frac{16}{3}$ sq. units.
4. **Solution:** $\frac{1}{2}$ sq. units.

5. **Solution:** 4 sq. units

6. **Solution:** 2 : 1.

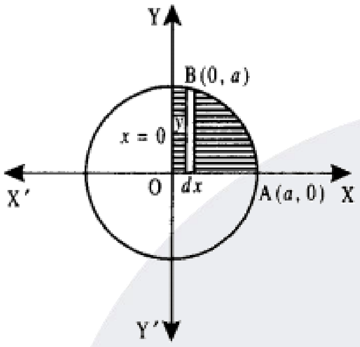
7. **Solution:** $\frac{1}{2}(\pi - 1)$ sq. units.

## Long Answer:

1. **Solution:**

The given circle is $x^2 + y^2 = a^2$ ............ (1)

This is a circle whose center is (0,0) and radius 'a'.




Area of the circle=4 x (area of the region OABO, bounded by the curve, x-axis and ordinates x = 0, x = a)

[ ∵ Circle is symmetrical about both the axes]

$= 4\int_0^a y dx$ [Taking vertical strips] o

$$=4\int_0^a \sqrt{a^2-x^2}dx$$

$\left[\because (1) \Rightarrow y = \pm\sqrt{a^2-x^2}\right.$ But region OABO lies in 1st quadrant, ∴ y is + ve]

$$=4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=4\left[\left\{\frac{a}{2}(0)+\frac{a^2}{2}\sin^{-1}(1)\right\}-\{0+0\}\right]$$

$$=4\left[\frac{a^2}{2}.\frac{\pi}{2}\right]=\pi a^2 \text{ sq. units.}$$

2. **Solution:**

We have:

y = x ...(l)

and $x^2 + y^2 = 32$ ...(2)

(1) is a st. line, passing through (0,0) and (2) is a circle with centre (0,0) and radius $4\sqrt{2}$ units. Solving (1) and (2) :

Putting the value of y from (1) in (2), we get:

$x^2 + x^2 = 32$

$2x^2 = 32$

$x^2 = 16$

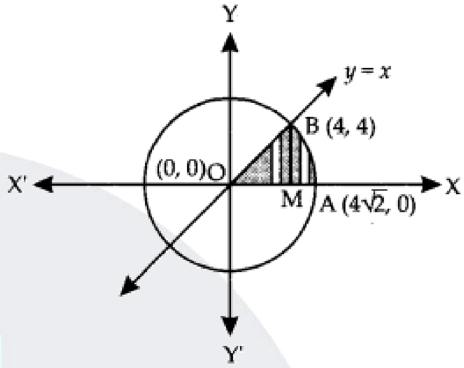
x = 4.

[∵ region lies in first quadrant]

Also, y = 4

Thus, the line (1) and the circle (2) meet each other at B (4,4), in the first quadrant.

Draw BM perp. to x – axis.




∴ Reqd. area = area of the region OMBO + area of the region BMAB ...(3)

Now, area of the region OMBO

$$=\int_0^4 y\,dx \quad \text{[Taking vertical strips]}$$

$$=\int_0^4 x\,dx=\left[\frac{x^2}{2}\right]_0^4=\frac{1}{2}(16-0)=8.$$

Again, area of the region BMAB

$$=\int_0^{4\sqrt{2}} y\,dx \quad \text{[Taking vertical strips]}$$

$$=\int_0^{4\sqrt{2}} \sqrt{32-x^2}\,dx$$

$\because y^2 = 32 - x^2 \Rightarrow y = \sqrt{32-x^2}$, taking +ve sign, as it lies in Ist quadrant]

$$=\int_0^{4\sqrt{2}} \sqrt{\left(4\sqrt{2}\right)^2-x^2}\,dx$$

$$=\left[\frac{x\sqrt{32-x^2}}{2}+\frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$=\left\{\frac{1}{2}4\sqrt{2}\times 0+\frac{32}{2}\sin^{-1}(1)\right\}-\left\{\frac{4}{2}\sqrt{32-16}+\frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right\}$$

$$=0+16\left(\frac{\pi}{2}\right)-\left(2\times 4+16\times\frac{\pi}{4}\right)$$

= 8π – (8 + 4π) = 4π – 8

∴ From (3),

Required area = 8 + (4π – 8) = 4π sq. units.

**3.** **Solution:**

The given curves are

$y = \sqrt{x}$ ............(1)

and 2y + 3 = x ...(2)

Solving (1) and (2), we get; $\sqrt{2y+3} = y$

Squaring, $2y + 3 = y^2$

$\Rightarrow y^2 2 - 2y - 3 = 0$

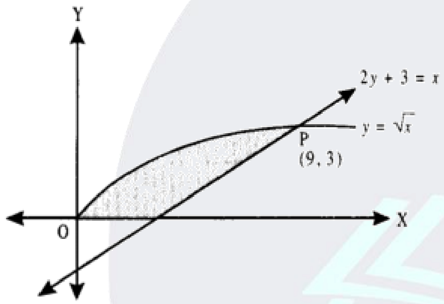
⇒ (y + 1)(y-3) = 0 ⇒ y = -1, 3

⇒ y = 3 [∵ y > 0]

Putting in (2),

x = 2(3) + 3 = 9.

Thus, (1) and (2) intersects at (9, 3).




$$\therefore \text{ Reqd. Area } = \int_0^3 (2y+3)dy - \int_0^3 y^2 dy$$

$$= \left[y^2 + 3y\right]_0^3 - \left[\frac{y^3}{3}\right]_0^3$$

$$= (9+9) - \left(\frac{27}{3}\right)$$

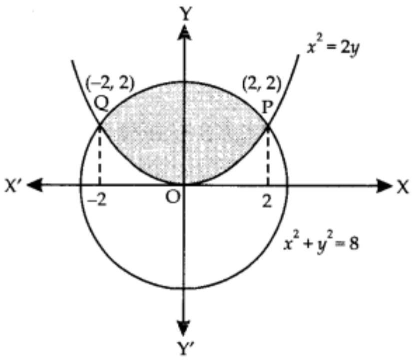
= 9 + 9 – 9 = 9 sq. units.

**4.** **Solution:**

The given curves are;

$x^2 + y^2 = 8$ ........... (1)

$x^2 = 2y$ ........... (2)




Solving (1) and (2):

$8 - y^2 = 2y$

$\Rightarrow y^2 + 2y - 8 = 0$

⇒ (y + 4) (y – 2) = 0

= y = -4,2

⇒ y = 2. [∵ y > 0]

Putting in (2), $x^2 = 4$

⇒ x = -2 or 2.

Thus, (1) and (2) intersect at P(2, 2) and Q(-2, 2).

$$\therefore \text{ Required area } = \int_{-2}^{2} \sqrt{8 - x^2}\,dx - \int_{-2}^{2} \frac{x^2}{2} dx$$

$$= \left[\int_0^2 \sqrt{\left(2\sqrt{2}\right)^2 - x^2}\,dx - \int_0^2 \frac{x^2}{2} dx\right]$$

$$= 2\left[\frac{x\sqrt{8-x^2}}{2} + \frac{8}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)\right]_0^2 - \frac{1}{3}\left[x^3\right]_0^2$$

$$= 2\left[2 + 4\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0\right] - \frac{1}{3}[8 - 0]$$

$$= 4 + 8\left(\frac{\pi}{4}\right) - \frac{8}{3} = \left(2\pi + \frac{4}{3}\right) sq.\ units.$$

## Case Study Answers:

**1.** **Answer:**

i. (b) x = ½

**Solution:**

We have, $(x - 1)^2 + y^2 = 1$

$$\Rightarrow\ y = \sqrt{1 - (x-1)^2}$$

$$\text{Also } x^2 + y^2 = 1 \Rightarrow\ y = \sqrt{1 - x^2}$$

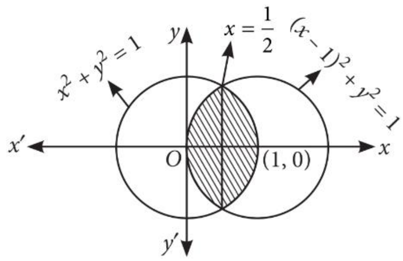
$$\sqrt{1-(x-1)^2} = \sqrt{1-x^2}$$

$$\Rightarrow (x-1)^2 = x^2$$

⇒ 2x = 1

$$\Rightarrow\ x = \frac{1}{2}$$

ii. (c)



iii. (a) $\frac{\pi}{6}-\frac{\sqrt{3}}{8}$

**Solution:**

$$\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{2}-1\right)\sqrt{1-\frac{1}{4}}+\frac{1}{2}+\sin^{-1}\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)-\frac{1}{2}\sin^{-1}$$

$$=\left[\frac{-1}{4}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\pi}{6}+0+\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{\sqrt{3}}{8}-\frac{\pi}{12}+\frac{\pi}{4}$$

$$=\frac{\pi}{6}-\frac{\sqrt{3}}{8}$$

iv. (c) $\frac{\pi}{6}-\frac{\sqrt{3}}{8}$

$$\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx=\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$=0+\frac{1}{2}\sin^{-1}(1)-\frac{1}{4}\sqrt{1-\frac{1}{4}}-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)$$

$$=\frac{\pi}{4}-\frac{\sqrt{3}}{8}-\frac{\pi}{12}=\frac{\pi}{6}-\frac{\sqrt{3}}{4}$$

v. (d)

**Solution:**

$$=2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right]$$

$$=\left[\frac{\pi}{6}-\frac{\sqrt{3}}{8}+\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right]$$

$$=2\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right]=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)sq.\,units$$

**2. Answer:**

**i. (a)** $y=\frac{3}{2}(x+1)$

**Solution:**

**Equation of line AB is**

$$y-0=\frac{3-0}{1+1}(x+1)\Rightarrow y=\frac{3}{2}(x+1)$$

ii. (c) $y=\frac{-1}{2}x+\frac{7}{2}$

**Solution:**

**Equation of line BC is** $y-3=\frac{2-3}{3-1}(x+1)$

$$\Rightarrow y=-\frac{1}{2}x+\frac{1}{2}+3\Rightarrow y=\frac{-1}{2}x+\frac{7}{2}$$

iii. (d) 8 sq. units

**Solution:**

Area of region ABCD = Area of ΔABE + Area of region BCDE

$$=\int_{-1}^{1}\frac{3}{2}(x+1)dx+\int_1^3\left(\frac{-1}{2}x+\frac{7}{2}\right)dx$$

$$=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}+\left[\frac{-x^2}{4}+\frac{7}{2}x\right]_1^3$$

$$=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]+\left[\frac{-9}{4}+\frac{21}{2}+\frac{1}{4}-\frac{7}{2}\right]$$

= 3 + 5 = 8 sq. units

iv. (a) 4 sq. units

**Solution:**

Equation of line AC is $y-0=\frac{2-0}{3+1}(x+1)$

$$\Rightarrow y=\frac{1}{2}(x+1)$$

∴ Area of ADC = $=\int_{-1}^{3}\frac{1}{2}(x+1)dx=\left[\frac{x^2}{4}+\frac{1}{2}x\right]_{-1}^{3}$

$$=\frac{9}{4}+\frac{3}{2}-\frac{1}{4}+\frac{1}{2}=4\,sq.units$$

v. (b) 4 sq. units

**Solution:**

Area of ΔABC = Area of region ABCD – Area of ΔACD = 8 – 4 = 4 sq. units

# Differential Equations 9

1. **Differential Equation:** An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
2. **Linear and non-linear differential equation:** A differential equation is said to be linear if unknown function (dependent variable) as its derivative which occurs in the equation, occur only in the first degree, and are not multiplied together. Otherwise, the differential equation is said to be non-linear.
3. **Order:** Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
4. **Degree:** Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
5. Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
6. **Solution:** A function which satisfies the given differential equation is called its solution.
7. **General Solution:** The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution.
8. **Particular Solution:** The solution free from arbitrary constants is called particular solution.
9. To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.
10. **Variable Separable method:** Variable separable method is used to solve such an equation in which variables can be separated completely i.e., terms containing y should remain with $dy$ and terms containing x should remain with dx.
11. A differential equation which can be expressed in the form $\frac{dy}{dx} f(x, y)$ $or$ $\frac{dx}{dy} g(x, y)$ where, f(x, y) and g (x, y) are homogenous functions of degree zero is called a homogeneous differential equation.
12. A differential equation of the form $\frac{dy}{dx} + Py$ = Q, where P and Q are constants or functions of x only is called a first order linear differential equation.

**Class : 12th Maths**
**Chapter- 9 : Differential Equations**

## Differential Equations

### Definition

An equation involving derivatives of the dependent variable with respect to independent variable (variables) is called a differential equation. If there is only one independent variation, then we call it as an ordinary differential equation. For eg: $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$.

### Variable Separation Method

It is used to solve such an equation in which variables can be separated completely. For eg: $y\,dx=x\,dy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$ Integrating both sides $\log x=\log y+\log c \Rightarrow \frac{x}{y}=c \Rightarrow x=cy$ ,is the solution.

### Order of a Differential Equation

It is the order of the highest order derivative occurring in the Differential Equation For eg: the order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

The order of a Differential Equations representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves. For eg: Let the family of curves be $y = mx$, $m =$ constant, then, $y'=m$

$y=y'x \Rightarrow y=\frac{dy}{dx}x \Rightarrow x\frac{dy}{dx}-y=0.$

### Degree of a Differential Equation

It is defined if the Differential Equations is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

For eg: the degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three

Order and degree (if defined) of a D.E. are always positive integers.

### Homogeneous Differential Equation

A Differential Equation which can be expressed in the form $\frac{dy}{dx}=f(x,y)$ or $\frac{dx}{dy}=g(x,y)$, where, $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree zero is called a homogenous Differential Equation

For $eg$ : $(x^2+xy)dy=(x^2+y^2)dx$

To solve this, we substitute $\frac{y}{x}=v \Rightarrow y=vx.$

### Linear Differential Equation

A Differential Equation of the form $\frac{dy}{dx}+Py=Q$, where $P$, $Q$ are constants or functions or '$x$' only is called a first order linear Differential Equations its solution is $ye^{\int P.dx}=\int Q.e^{\int P.dx}dx+c$. For $eg$ : $\frac{dy}{dx}+3y=2x$ has solution $ye^{\int 3.dx}=\int 2x.e^{\int 3.dx}dx+c \Rightarrow ye^{3x}=2\int xe^{3x}+c.$

### Solution of a Differential Equation

A function which satisfies the given Differential Equation is called its solution. The solution which contains as many arbitrary constants as the order of the D.E. is called a general solution and the solution free from arbitrary constants is called particular solution.

For eg: $y=e^x+1$ is a solution of $y''-y'=0$.

Since $y'=e^x$ and $y''=e^x \Rightarrow y''-y'=e^x-e^x=0.$

### Formation of Differential Equation

To form a Differential Equation from a given function, we differentiate the function successively as many times as the no. of arbitrary constants in the given function, and then eliminate the arbitrary constants.

For eg: Let the function be $y=ax+b$, then we have to differentiate it two times, since there are 2 arbitrary constants $a$ and $b$. $\therefore y'=a \Rightarrow y''=0$. Thus $y''=0$ is the required Differential Equation.

# Important Questions

## Multiple Choice questions-

1. The degree of the differential equation:

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0 \text{ is}$$

(a) 3
(b) 2
(c) 1
(d) not defined.

2. The order of the differential equation:

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0 \text{ is}$$

(a) 2
(b) 1
(c) 0
(d) not defined.

3. The number of arbitrary constants in the general solution of a differential equation of fourth order is:

(a) 0
(b) 2
(c) 3
(d) 4.

4. The number of arbitrary constants in the particular solution of a differential equation of third order is:

(a) 3
(b) 2
(c) 1
(d) 0.

5. Which of the following differential equations has $y = c_1 e^x + c_2 e^{-x}$ as the general solution?

(a) $\frac{d^2y}{dx^2} + y = 0$
(b) $\frac{d^2y}{dx^2} - y = 0$
(c) $\frac{d^2y}{dx^2} + 1 = 0$
(d) $\frac{d^2y}{dx^2} - 1 = 0$

6. Which of the following differential equations has y = x as one of its particular solutions?

(a) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$
(b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$
(c) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$
(d) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

7. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(a) $e^x + e^{-y} = c$
(b) $e^x + e^y = c$
(c) $e^{-x} + e^y = c$
(d) $e^{-x} + e^{-y} = c$.

8. Which of the following differential equations cannot be solved, using variable separable method?

(a) $\frac{dy}{dx} + e^{x+y} + e^{-x+y}$
(b) $(y^2 - 2xy)\, dx = (x^2 - 2xy)\, dy$
(c) $xy\frac{dy}{dx} = 1 + x + y + xy$
(d) $\frac{dy}{dx} + y = 2$.

9. A homogeneous differential equation of the form $\frac{dy}{dx}=h\left(\frac{x}{y}\right)$ can be solved by making the substitution.

(a) $y = vx$
(b) $v = yx$
(c) $x = vy$
(d) $x = v$

10. Which of the following is a homogeneous differential equation?

(a) $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$
(b) $xy\, dx - (x^3 + y^2)dy = Q$
(c) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$
(d) $y^2\, dx + (x^2 - xy - y^2)dy = 0$.

## Very Short Questions:

1. Find the order and the degree of the differential equation: $x^2\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^4$

2. Determine the order and the degree of the differential equation: $\left(\frac{dy}{dx}\right)^3 + 2y\frac{d^2y}{dx^2} = 0$

3. Form the differential equation representing the family of curves: y = b (x + a), where « and b are arbitrary constants.

4. Write the general solution of differential equation:

$\frac{dy}{dx} = e^{x+y}$

5. Find the integrating factor of the differential equation:

$y\frac{dy}{dx} - 2x = y^3e^{-y}$

6. Form the differential equation representing the family of curves y = a sin (3x – b), where a and b are arbitrary constants.

## Short Questions:

1. Determine the order and the degree of the differential equation:
2. Form the differential equation representing the family of curves: $y = e^{2x}(a + bx)$, where 'a' and 'h' are arbitrary constants.
3. Solve the following differential equation:
$\frac{dy}{dx} + y = \cos x - \sin x$
4. Solve the following differential equation:
$\frac{dx}{dy} + x = (\tan y + \sec 2y)$.

## Long Questions:

1. Find the area enclosed by the circle:
$x^2 + y^2 = a^2$.
2. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$.
3. Find the area bounded by the curves $y = \sqrt{x}$, 2y + 3 = Y and Y-axis.
4. Find the area of region:
$\{(x,y): x^2 + y^2 < 8, x^2 < 2y\}$.

## Case Study Questions:

**1.** If the equation is of the form $\frac{dy}{dx} = py = Q$, where P, Q are functions of x, then the solution of the differential equation is given by $ye^{\int pdx} = \int Q e^{\int pdx} dx + c$, where $e^{\int pdx}$ is called the integrating factor (I.F.).

Based on the above information, answer the following questions.

i. The integrating factor of the differential equation $\sin x \frac{dy}{dx} + 2y \cos x = 1$ is $(\sin x)^{\lambda}$, where λ =
(a) 0
(b) 1
(c) 2
(d) 3

ii. Integrating factor of the differential equation $(1 - x^2)\frac{dy}{d} - xy = 1$ is
(a) -x
(b) $\frac{x}{1+x^2}$
(c) $\sqrt{1-x^2}$
(d) $\frac{1}{2}\log(1-x^2)$

iii. The solution of $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$ is
(a) $y = e^x(x-1)$
(b) $y = xe^{-x}$
(c) $y = xe^{-x} + 1$
(d) $y = (x+1)e^{-x}$

iv. General solution of $\frac{dy}{dx} + y\tan x = \sec x$ is:
(a) $y\sec y = \tan x + c$
(b) $y\tan x = \sec x + c$
(c) $\tan x = y\tan x + c$
(d) $x\sec x = \tan y + c$

v. The integrating factor of differential equation $\frac{dy}{dx} - 3y = \sin 2x$ is:
(a) $e^{3x}$
(b) $e^{-2x}$
(c) $e^{-3x}$
(d) $xe^{-3x}$

**2.** If the equation is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ or $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, where $f(x, y)$, $g(x, y)$ are homogeneous functions of the same degree in x and y, then put y = vx and $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$, so that the dependent variable y is changed to another variable v and then apply variable separable method.

Based on the above information, answer the following questions.

i. The general solution of $x^2\frac{dy}{dx} = x^2 + xy + y^2$ is:
(a) $\tan^{-1}\frac{x}{y} = \log|x| + c$
(b) $\tan^{-1}\frac{y}{x} = \log|x| + c$
(c) $y = x\log|x| + c$

(d) $x = y\log|y| + c$

ii. Solution of the differential equation $2xy\dfrac{dy}{dx} = x^2 + 3y^2$ is:

(a) $x^3 + y^2 = cx^2$

(b) $\dfrac{x^2}{2} + \dfrac{y^3}{3} = y^2 + c$

(c) $x^2 + y^3 = cx^2$

(d) $x^2 + y^2 = cx^3$

iii. General solution of the differential equation $(x^2 + 3xy + y^2)\,dx - x^2dy = 0$ is:

(a) $\dfrac{x+y}{y} - \log x = c$

(b) $\dfrac{x+y}{y} + \log x = c$

(c) $\dfrac{x}{x+y} - \log x = c$

(d) $\dfrac{x}{x+y} + \log x = c$

iv. General solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x}\left\{\log\left(\dfrac{y}{x}\right) + 1\right\}$ is:

(a) log (*xy*) = *c*

(b) log *y* = *cx*

(c) $\log\dfrac{y}{x} = cx$

(d) log *x* = *cy*

v. Solution of the differential equation $\left(x\dfrac{dy}{dx} - y\right)e^{\frac{y}{x}} = x^2 \cos x$ is:

(a) $e^{\frac{y}{x}} - \sin x = c$

(b) $e^{\frac{y}{x}} + \sin x = c$

(c) $e^{\frac{-y}{x}} - \sin x = c$

(d) $e^{\frac{-y}{x}} + \sin x = c$

# Answer Key

## Multiple Choice questions-

1. **Answer:** (a) 3
2. **Answer:** (a) 2
3. **Answer:** (d) 4.
4. **Answer:** (d) 0.
5. **Answer:** (b) $\frac{d^2y}{dx^2} - y = 0$
6. **Answer:** (c) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$
7. **Answer:** (a) $e^x + e^{-y} = c$
8. **Answer:** (b) $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$
9. **Answer:** (c) x = vy
10. **Answer:** (d) $y^2\,dx + (x^2 - xy - y^2)dy = 0$.

## Very Short Answer:

1. **Solution:** Here, order = 2 and degree = 1.
2. **Solution:** Order = 2 and Degree = 1.
3. **Solution:**

We have: y= b(x + a) ...(1)

Diff. w.r.t. x, b.

Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = 0$,

which is the reqd. differential equation.

4. **Solution:**

We have: $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow e^{-y}\,dy = e^x\,dx$ [Variables Separable

Integrating, $\int e^{-y}dy + c = \int e^x dx$

$\Rightarrow -e^{-y} + c = e^x$

$\Rightarrow e^x + e^{-y} = c.$

5. **Solution:**

The given equation can be written as.

$$\frac{dy}{dx} - \frac{2x}{y} = y^2e^{-y}$$

$$\therefore I.F. = e^{-\int \frac{2}{y}dy}$$

$$= e^{-2\log|y|} = e^{\log\frac{1}{y^2}} = \frac{1}{y^2}$$

6. **Solution:**

We have: y – a sin (3x – b) ...(1)

Diff. W.r.t y $\frac{dy}{dx}$ = a cos (3x – b) .3

= 3a cos (3x – b)

$\frac{d^2y}{dx^2}$ = -3a sin (3x – b) 3

= -9a sin (3x – b)

= -9y [Using (1)]

$\frac{d^2y}{dx^2}$ + 9y = 0, m

which in the reqd. differential equation.

## Short Answer:

1. **Solution:** Order = 2 and Degree = 1.

2. **Solution:**

We have: y = $e^{2x}$ (a + bx) ...(1)

Diff. w.r.t. x, $\frac{dy}{dx}$ = $e^{2x}$ (b) + 2e2x (a + bx)

⇒ $\frac{dy}{dx}$ = $be^{2x}$ + 2y ............. (2)

Again diff. w.r.t. x,

$\frac{d^2y}{dx^2}$ = $2be^{2x}$ + $2^{2x}$

$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - 2y\right) + \frac{dy}{dx}$

[Using (2)]

Hence, $\frac{d^2y}{dx^2}$ -4 $\frac{dy}{dx}$ + 4y = 0, which is the reqd. differential equation.

3. **Solution:**

The given differential equation is:

$\frac{dy}{dx}$ + y = cos x – sin x dx Linear Equation

∴ I.F. = $e^{\int 1dx}$ = ex

The solution is :

y.$e^x$ = ∫ (cos x — sin x) $e^x$ dx + C

⇒ y.$e^x$ = $e^x$ cos x + C

or y = cos x + C $e^{-x}$

4. **Solution:**

The given differential equation is:

$\frac{dx}{dy}$ + x = (tany + $sec^2$y).

Linear Equation

∵ I.F. = ∫ldy = ey

∴ The solution is:

x. ey = ∫ ey (tan y + $sec^2$ y)dy + c

⇒ x. ey = ey tan y + c

= x = tan y + c $e^{-y}$, which is the reqd. solution.

## Long Answer:

1. **Solution:**

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\frac{y^2}{x^2} - 1}{\frac{2y}{x}}$$

Put $\frac{y}{x} = v$

$\Rightarrow y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$

$\Rightarrow \frac{x\,dv}{dx} = -\frac{(1+v^2)}{2v}$

$\Rightarrow \int \frac{dx}{x} = -\int \frac{2v\,dv}{1+v^2}$

log x = -log (1 + $v^2$) + log C

x(1 + $v^2$) = C

$x\left(1 + \frac{y^2}{x^2}\right) = C$

$x^2 + y^2$ = C.

2. **Solution:**

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$$

I.F. = $e^{\int \frac{2x}{1+x^2}} + e^{\log(1+x^2)} = (1+x^2)$

Solution is $y(1+x^2) = \int \frac{1}{1+x^2}dx$

= $\tan^{-1}$ x + C

When y = 0, x = 1,

then 0 = $\frac{\pi}{4}$ + C

C = $\frac{\pi}{4}$

∴ y (1 + $x^2$) = $\tan^{-1}$ x – $\frac{\pi}{4}$

i.e., $y = \frac{\tan^{-1} x}{1+x^2} - \frac{\pi}{4(1+x^2)}$

3. **Solution:**

We have: y = $ae^{bx+5}$ + 5 ...(1)

Diff. w.r.t. x, $\frac{dy}{dx}$ = $ae^{bx+5}$. (b)

$\frac{dy}{dx}$ = dy ......(2) [Using (1)]]

Again diff. w.r.t x.,

$\frac{d^2y}{dx^2}$ = b $\frac{dy}{dx}$ .........(3)

Dividing (3) by (2),

$$\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} = \frac{\frac{dy}{dx}}{y}$$

$$\Rightarrow y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation.

4. **Solution:**

The given differential equation is:

$x\,dx - ye^y\sqrt{1+x^2}\,dy = 0$

$\Rightarrow \frac{x\,dx}{\sqrt{1+x^2}} - ye\,dy = 0$ [Variables Separable]

Integrating, $\int \frac{x\,dx}{\sqrt{1+x^2}} - \int ye^y\,dy = c$ ....(1)

Now, $\int \frac{x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int (1+x^2)^{-1/2}(2x)\,dx$

$= \frac{1}{2}\frac{(1+x^2)^{1/2}}{1/2} = \sqrt{1+x^2}$

And, $\int y\,e^y dy = y\cdot e^y - \int (1)\,e^y dy$

[Integrating by parts]

$= ye^y - e^y$.

$\therefore$ From (1), $\sqrt{1+x^2} - (ye^y - e^y) = c$

$\Rightarrow \sqrt{1+x^2} = c + e^y(y-1)$ ....(2)

When x = 0, y = 1, $\therefore$ 1 = c + c(0) $\Rightarrow$ c = 1.

Putting in (2), $\sqrt{1+x^2} = 1 + e^y(y-1)$,

which is the reqd. particular solution.

## Case Study Answers:

**1. Answer :**

i. (c) 2

**Solution:**

The given differential equation can be written as $\frac{dy}{dx} + 2y\cot x = \operatorname{cosec} x$

$\therefore\ I.F. = e^{\int 2\cot x\,dx} = e^{2\log|\sin x|} = (\sin x)^2$

$\therefore\ \lambda = 2$

ii. (c) $\sqrt{1-x^2}$

**Solution:**

We have, $(1-x^2)\frac{dy}{dx} - xy = 1$

$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}\cdot y = \frac{1}{1-x^2}$

$\therefore\ I.F. = e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}dx}$

$= e^{\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{\frac{1}{2}}} = \sqrt{1-x^2}$

iii. (b) $y = xe^{-x}$

**Solution:**

We have, $\frac{dy}{dx} + y = e^{-x}$

It is a linear differential equation with

$I.F. = e^{\int dx} = e^x$

Now, solution is $y\cdot e^x = \int e^{-x}dx + c$

$\Rightarrow ye^x = \int dx + c$

$\Rightarrow ye^x = x + c$

$\Rightarrow y = xe^{-x} + ce^{-x}$

$\because\ y(0) = 0 \Rightarrow c = 0$

$\therefore\ y = xe^{-x}$

iv. (c) $y\sec y = \tan x + c$

**Solution:**

We have, $\frac{dy}{dx} + y\tan x = \sec x$

It is a linear differential equation with,

$I.F. = e^{\int \tan x\,dx} = e^{\log|\sec x|} = \sec x$

Now, solution is $y\sec x = \int \sec^2 x\,dx + c$

$\Rightarrow y\sec x = \tan x + c$

v. (c) $e^{-3x}$

**Solution:**

We have, $\frac{dy}{dx} 3y = \sin 2x$

It is a linear differential equation with,

$I.F. = e^{\int -3dx} = e^{-3x}$

**2. Answer :**

i. (b) $\tan^{-1}\frac{y}{x} = \log|x| + c$

**Solution:**

We have, $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore\ v + x\frac{dv}{dx} = \frac{x^2 + x\times vx + v^2x^2}{x^2} = 1 + v + v^2$

$\Rightarrow x\frac{dv}{dx} = 1 + v^2 \Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x} + c$

$\Rightarrow \tan^{-1} v = \log|x| + c$

$\Rightarrow \tan^{-1}\frac{y}{x} = \log|x| + c$

ii. (d) $x^2 + y^2 = cx^3$

**Solution:**

We have,

$2xy\frac{dy}{dx} = x^2 + 3y^2$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$$

Put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore\ v+x\frac{dv}{dx}=\frac{x^2+3v^2x^2}{2vx^2}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+3v^2}{2v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{2v}$$

$$\Rightarrow \int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}+\log c$$

$$\Rightarrow \log\left|1+v^2\right|=\log|x|+\log|c|$$

$$\Rightarrow \log\left|v^2+1\right|=\log|xc|$$

$$\Rightarrow v^2+1=xc \Rightarrow \frac{y^2}{x^2}+1=xc$$

$$\Rightarrow x^2+y^2=x^3c$$

iii. (d) $\frac{x}{x+y}+\log x=c$

**Solution:**

We have,

$(x^2 + 3xy + y^2)\ dx - x^2\ dy = 0$

$$\Rightarrow \frac{x^2+3xy+y^2}{x^2}=\frac{dy}{dx}$$

Put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore\ \frac{x^2+3x^2v+x^2v^2}{x^2}=\left(v+x\frac{dv}{dx}\right)$$

$$\Rightarrow 1+3v+v^2=v+x\frac{dv}{dx}$$

$$\Rightarrow 1+2v+v^2=x\frac{dv}{dx}$$

$$\Rightarrow \int\frac{dx}{x}-\int(v+1)^{-2}=dv=c$$

$$\log x+\frac{1}{v+1}=c$$

$$\Rightarrow \log x+\frac{x}{x+y}=c$$

iv. (c) $\log\frac{y}{x}=cx$

**Solution:**

We have, $\frac{dy}{dx}=\frac{y}{x}\left\{\log\left(\frac{y}{x}\right)+1\right\}$

Put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore\ v+x\frac{dv}{dx}=v\{\log(v+1)\}$$

$$\Rightarrow x\frac{dv}{dx}=v\log v$$

$$\Rightarrow \int\frac{dv}{v\log v}=\int\frac{dx}{x} \Rightarrow \log|\log v|=\log|x|+\log|c|$$

$$\Rightarrow \log\left(\frac{y}{x}\right)=cx$$

v. (a) $e^{\frac{y}{x}}-\sin x=c$

Solution:

We have, $\left(x\frac{dy}{dx}-y\right)e^{\frac{y}{x}}=x^2\cos x$

$$\Rightarrow \left(\frac{dy}{dx}-\frac{y}{x}\right)e^{\frac{y}{x}}=x\cos x$$

$$\Rightarrow xe^{v}\frac{dv}{dx}=x\cos x$$

$$\Rightarrow \int e^v dv=\int\cos x\,dx$$

$$\Rightarrow e^v=\sin x+c$$

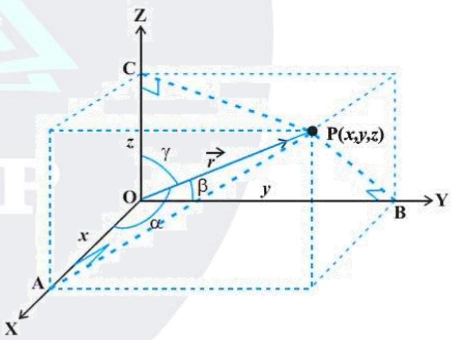
$$\Rightarrow e^{\frac{y}{x}}-\sin x=c$$

# Vector Algebra 10

## Top Concepts

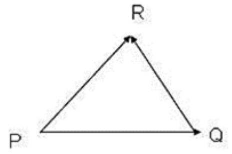
1. A quantity which has magnitude as well as direction is called a vector.
2. A directed line segment is called a vector.
   The point X from where the vector starts is called the initial point and the point Y where it ends is called the terminal point.


3. For vector $\underline{XY}$, magnitude = distance between X and Y, denoted by $|\underline{XY}|$, and is greater than or equal to zero.
4. The distance between the initial point and the terminal point is called the magnitude of the vector.
5. The position vector of point $P = (x_1. y_1. z_1)$ with respect to the origin is given by $\underline{OP} = \vec{r} = \sqrt{x_1^2 + y_1^2 + z_1^2}$.
6. If the position vector $\underline{OP}$ of a point P makes angles $\alpha$, β and $\gamma$ with the x, y and z axes, respectively, then $\alpha$, β and $\gamma$ are called the direction angles and cos $\alpha$, cos β and cos $\gamma$ are called the direction cosines of the position vector $\underline{OP}$.
7. λ = cos $\alpha$, m = cos β and n = cos $\gamma$ are called the direction cosines of $\vec{r}$.
8. The numbers Ir, mr and nr, proportional to I, m and n, are called the direction ratios of the vector $\vec{r}$ and are denoted by a, b and c.
   In general, $l^2 + m^2 + n^2 = 1$, but $a^2 + b^2 + c^2 \neq 1$.
9. Vectors can be classified on the basis of position and magnitude. On the basis of magnitude, vectors are classified as zero vectors and unit vectors. On the basis of position, vectors are classified as co- initial vectors, parallel vectors, free vectors and collinear vectors.



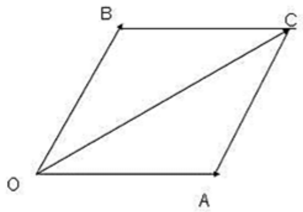

10. A zero vector is a vector whose initial and terminal points coincide and is denoted by $\vec{0}$ . $\vec{0}$ is called the additive identity.
11. A unit vector has a magnitude equal to 1. A unit vector in the direction of the given vector $\vec{a}$ is denoted by $\hat{a}$.
12. For a given vector $\vec{a}$ , the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$.
13. Co-initial vectors are vectors with the same initial point.
14. Collinear vectors are parallel to the same line irrespective of their magnitudes and directions.
15. Two vectors are said to be parallel if they are non-zero scalar multiples of one another.
16. Equal vectors, as the name suggests, are vectors which have the same magnitude and direction irrespective of their initial points.
17. The negative vector of a given vector $\vec{a}$ is a vector which has the same magnitude as $\vec{a}$ but its direction is the opposite of $\vec{a}$ .
18. A system of vectors is said to be coplanar, if the vectors lie in the same plane.
19. Reciprocal of a vector: A vector with the same direction as that of the given vector $\vec{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\underline{a^{-1}}$.
   Thus, if $\vec{a}$ is a vector with magnitude a, then $|\underline{a^{-1}}| = \frac{1}{a}$.

20. A vector whose initial position is not fixed is called a free vector.
21. Two vectors can be added using the triangle law and parallelogram law of vector addition.
22. **Triangle Law of Vector Addition:** Suppose two vectors are represented by two sides of a triangle in sequence, then the third closing side of the triangle represents the sum of the two vectors.




$$\overline{PQ}+\overline{QR}=\overline{PR}$$

23. **Parallelogram Law of Vector Addition:** If two vectors $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a}$ + $\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram.




$$\overline{OA}+\overline{OB}=\overline{OC}$$

24. Vector addition is both commutative as well as associative.

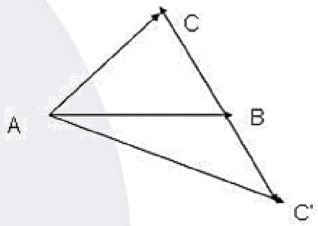
Thus, (i) Commutative: For any two vectors $\vec{a}$ and $\vec{b}$, we have

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) Associativity: For any three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, we have

$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

25. Difference of vectors: To subtract a vector $\underline{BC}$ from vector $\underline{AB}$, its negative is added to $\underline{AB}$.




$\overline{BC'}=-\overline{BC}$

$\overline{AB}+\overline{BC'}=\overline{AC'}$

$\Rightarrow \overline{AB}-\overline{BC}=\overline{AC'}$

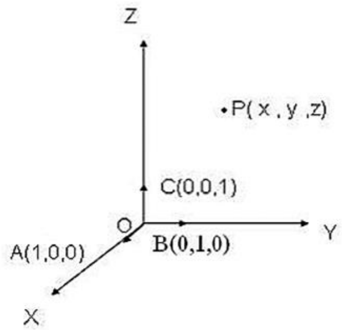
26. For any two vectors $\vec{a}$ and $\vec{b}$, we have $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$.
27. For any two vectors $\vec{a}$ and $\vec{b}$, we have $|\vec{a}-\vec{b}| \le |\vec{a}|+|\vec{b}|$.
28. For any two vectors $\vec{a}$ and $\vec{b}$, we have $|\vec{a}-\vec{b}| \ge ||\vec{a}|-|\vec{b}||$.
29. If $\vec{a}$ is any vector and k is any scalar, then the scalar product of $\vec{a}$ and k is k$\vec{a}$. k $\vec{a}$ is also a vector, collinear to the vector $\vec{a}$.

k > 0 ⇒ k $\vec{a}$has the same direction as $\vec{a}$.

k < 0 ⇒ k $\vec{a}$ has the opposite direction as $\vec{a}$.

Magnitude of k $\vec{a}$ is |k| times the magnitude of vector k $\vec{a}$.

30. Unit vectors along OX, OY and OZ are denoted by $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$, respectively.




Vector $\overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is called the component form of vector r.

Here, x, y and z are called the scalar components of $\vec{r}$ in the directions of and $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$, and

$x\hat{\imath}$, $y\hat{\jmath}$ and $z\hat{k}$ are called the vector components of the vector r along the respective axes.

31. Two vectors $\vec{a}$ and $\vec{b}$ are collinear ⇔ $\vec{b}$ = k $\vec{a}$, where k is a non-zero scalar. Vectors $\vec{a}$ and k $\vec{a}$ are always collinear.
32. If $\vec{a}$ and $\vec{b}$ are equal, then $|\vec{a}| = |\vec{b}|$.
33. If P($x_1, y_1, z_1$) and Q ($x_2, y_2, z_2$) are any two points, then the vector joining P and Q is $\overrightarrow{PQ}$ = position vector of Q — position vector of P, i.e., $\overrightarrow{PQ}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$.
34. Components of a vector in two dimensions: If Q is a point X(a, b), then

(i) $\overrightarrow{OQ}=a\hat{i}+b\hat{j}$

(ii) $|\overrightarrow{OQ}| = \sqrt{a^2 + b^2}$

(iii) The components of $\overrightarrow{OO}$ along the x-axis is a vector $a\hat{i}$ whose magnitude is |a| and whose direction is along OX or OX' as a is positive or negative.

(iv) The components of $\overrightarrow{OQ}$ along the y-axis is a vector $b\hat{j}$ whose magnitude is |b| and whose direction is along OY or OY' as b is positive or negative.

35. Components of a vector in three dimensions: If Q is a point X (a, b, c), then

(i) $\overrightarrow{OQ} = a\hat{i} + b\hat{j} + c\hat{k}$

(ii) $|\overrightarrow{OQ}| = \sqrt{a^2 + b^2 + c^2}$

(iii) The components of $\overrightarrow{OQ}$ along the x-axis is a vector $a\hat{i}$ whose magnitude is |a| and whose direction is along OX or OX' as a is positive or negative.

(iv) The components of $\overrightarrow{OQ}$ along the y-axis is a vector $b\hat{j}$ whose magnitude is |b| and whose direction is along OY or OY' as b is positive or negative.

(v) The components of $\overrightarrow{OQ}$ along the z-axis is a vector $c\hat{k}$ whose magnitude is |c| and whose direction is along OZ or OZ' as c is positive or negative.

36. Collinearity of vectors: If $\vec{a}$ and $\vec{b}$ are two collinear or parallel vectors, then there exists a scalar k such that $\vec{a} = k\vec{b}$.

37. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists scalars x and y, not both zero, such that $x\vec{a} + y\vec{b} = \vec{0}$.

38. If $\vec{a}$ and $\vec{b}$ are any two non-zero non-collinear vectors and x and y are scalars, then $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$.

39. Three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ care collinear if and only if there exists three scalars x, y and z, not all zero simultaneously, such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$ and $x + y + z = 0$.

40. Let $\vec{a}$ and $\vec{b}$ be two given non-zero non-collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r} = m\vec{a} + n\vec{b}$ for scalars m and n.

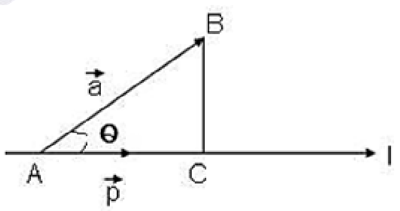
41. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if and only if

$l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$, where I, m and n are scalars not all zero simultaneously.

42. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three non-zero non-coplanar vectors, and x, y and z are scalars, then $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0$.

43. Geometrical interpretation of the scalar product: scalar product of two vectors $\vec{a}$ and $\vec{b}$ is the product of modulus of either $\vec{a}$ or $\vec{b}$, and the projection of the other in its direction.

44. Projection of vector AB, making an angle of $\theta$ with the line L, on line L is vector $\vec{P} = |\overrightarrow{AB}| \cos\theta$.




45. Properties of a scalar product:

(i) Scalar product is commutative, i.e., $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(ii) Distributive property of a scalar product over addition:

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$(\vec{b} \cdot \vec{c}) \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}$

(iii) If the scalar product $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

(iv) If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then the scalar product $\vec{a} \cdot \vec{b} = 0$.

(v) If $\vec{a}$ is any vector, then $\vec{a} \cdot \vec{a}$ $|\vec{a}|^{-2}$.

(vi) If $\vec{a}$ and $\vec{b}$ are two vectors, then $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot m\vec{b}$, where m is a scalar.

(vii) If $\vec{a}$ and $\vec{b}$ are two vectors, then $m\vec{a} \cdot n\vec{b} = mn(\vec{a} \cdot \vec{b}) = (mn\vec{a}) \cdot \vec{b} = \vec{a} \cdot mn\vec{b}$, where $m$ and $n$ are scalars.

(viii) If $\vec{a}$ and $\vec{b}$ are two vectors, then

$$\vec{a}\cdot(-\vec{b})=-(\vec{a}\cdot\vec{b})=(-\vec{a})\cdot\vec{b}$$

$$(-\vec{a})\cdot(-\vec{b})=\vec{a}\cdot\vec{b}$$

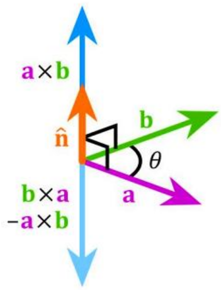
(ix) If $\vec{a}$ and $\vec{b}$ are two vectors, then

$$|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}$$

$$|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2$$

46. The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\times\vec{b}$ is defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\ sin\ sin\ \theta\hat{n}$, where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. Here, $\vec{a}$, $\vec{b}$ and $\hat{n}$, form a right-handed system.




47. Properties of a vector product:

(i) $\vec{a}\times\vec{b}$ is a vector.

(ii) Vector product is not commutative.

$$\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$$

(iii) If $\vec{a}$ and $\vec{b}$ are non-zero vectors, then $\vec{a}\times\vec{b}=0$ if and only if $\vec{a}$ and $\vec{b}$ are collinear, i.e. $\vec{a}\times\vec{b}=0\Leftrightarrow\vec{a}\,||\,\vec{b}$, here either θ = 0 or θ = π.

(iv) If $\theta=\frac{\pi}{2}$, then $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|$.

(v) If $\vec{a}$ and $\vec{b}$ are two vectors, λ is a scalar, then $\lambda\vec{a}\times\vec{b}=\lambda(\vec{a}\times\vec{b})=\vec{a}\times\lambda\vec{b}$

(vi) If $\vec{a}$ and $\vec{b}$ are two vectors, and λ and μ are scalars, then $\lambda\vec{a}\times\mu\vec{b}=\lambda\mu(\vec{a}\times\vec{b})=\lambda(\vec{a}\times\mu\vec{b})=\mu(\lambda\vec{a}\times\vec{b})$

(vii) Vector product is distributive over addition.

If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors, then

(i) $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$

(ii) $(\vec{b}+\vec{c})\times\vec{a}=(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})$

(viii) Vector product is distributive over subtraction.

If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors, then

(i) $\vec{a}\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})$

(ii) $(\vec{b}-\vec{c})\times\vec{a}=(\vec{b}\times\vec{a})-(\vec{c}\times\vec{a})$

(ix) If we have two vectors $\vec{a}$ and $\vec{b}$ given in component form as,

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$

(x) For unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$,

$$\hat{i}\times\hat{j}=0;\ \hat{j}\times\hat{j}=0;\ \hat{k}\times\hat{k}=0$$

$$\hat{i}\times\hat{j}=\hat{k};\ \hat{j}\times\hat{k}=\hat{i};\ \hat{k}\times\hat{i}=\hat{j}$$

$$\hat{j}\times\hat{i}=-\hat{k};\ \hat{k}\times\hat{j}=-\hat{i};\ \hat{i}\times\hat{k}=-\hat{j}$$

(xi) $\vec{a}\times\vec{b}=\vec{0}$ as $\theta=0 \therefore \sin\theta=0$

$\vec{a}\times(-\vec{a})=\vec{0}$ as $\theta=\pi \therefore \sin\theta=0$

$\vec{a}\perp\vec{b} \Rightarrow \theta=\frac{\pi}{2} \Rightarrow \sin\theta=1 \Rightarrow \vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\hat{n}$

(xii) The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \text{ or } \theta=\sin^{-1}\left(\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right)$$

48. Let $\vec{a}$ and $\vec{b}$ be two vectors. Then the vectors perpendicular to $\vec{a}$ and $\vec{b}$ with magnitude 'k' are given by $\frac{\pm k(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

49. The area of a parallelogram is equal to the modulus of the cross product of the vectors representing its adjacent sides.

$A=|\vec{a}\times\vec{b}|$

50. The area of a parallelogram with diagonals $\vec{c}$ and $\vec{d}$ is $\frac{1}{2}|\vec{c}\times\vec{d}|$.

51. The area of a triangle is equal to half of the modulus of the cross product of the vectors representing its adjacent sides.

$A=\frac{1}{2}|\vec{a}\times\vec{b}|$

52. The area of Δ ABC is $\frac{1}{2}\times|\vec{AB}\times\vec{AC}|$ or $\frac{1}{2}|\vec{BC}\times\vec{BA}|$ or $\frac{1}{2}|\vec{CB}\times\vec{CA}|$

53. The area of a Δ ABC with position vectors of the vertices A, B and C is area of $\Delta ABC=\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$.

54. The length of the perpendicular from $C$ on $AB=\frac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{|\vec{a}-\vec{b}|}$.

55. The length of the perpendicular from $A$ on $BC=\frac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{|\vec{b}-\vec{c}|}$.

56. The length of the perpendicular from $B$ on $AC=\frac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{|\vec{c}-\vec{a}|}$.

57. The area of a quadrilateral ABCD is $\frac{1}{2}|\vec{AC}\times\vec{BD}|$, where, $\vec{AC}$ and $\vec{BD}$ are its diagonals.

58. The vector sum of the sides of a triangle taken in order is zero.

## Top Formulae

1. Properties of addition of vectors

(i) Vector addition is commutative

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) Vector addition is associative.

$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

(iii) $\vec{0}$ is an additive identity for vector addition.

$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$

2. The position vector of the point C which divides AB internally in the ratio m : n is given by $\overrightarrow{OC}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$.

3. The position vector of the point C which divides AB externally in the ratio m : n is given by $\overrightarrow{OC}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$.

4. Linear combination of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is of the form $\vec{r}$ = x$\vec{a}$ + y$\vec{b}$ + z$\vec{c}$, where x, y and z are the scalars.

5. The position vector of the centroid is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the triangle.

6. Magnitude or length of the vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is $|\vec{r}|=\sqrt{x^2+y^2+z^2}$.

7. Vector addition in component form: Given $\vec{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r_2}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, then $\vec{r_1}+\vec{r_2}=(x_1+x_2)\hat{i}+(y_1+y_2)\hat{j}+(z_1+z_2)\hat{k}$.

8. Difference of vectors: Given $\vec{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r_2}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, then $\vec{r_1}-\vec{r_2}=(x_1-x_2)\hat{i}+(y_1-y_2)\hat{j}+(z_1-z_2)\hat{k}$.

9. Equal vectors: Given $\vec{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r_2}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, then $\vec{r_1}=\vec{r_2}$ if and only if $x_1=x_2;\ y_1=y_2;\ z_1=z_2$

10. Multiplication of $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ with scalar $k$ is given by $k\vec{r}=(kx)\hat{i}+(ky)\hat{j}+(kx)\hat{k}$.

11. For any vector, $\vec{r}$ in component form $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then $x$, $y$ and $z$ are the direction ratios of $\vec{r}$ and $\dfrac{x}{\sqrt{x^2y^2+z^2}},\dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $and$ $\dfrac{z}{\sqrt{x^2+y^2+z^2}}$ are its direction cosines.

12. The direction ratios of the line segment joining points ($x_1$, $y_1$, $z_1$) and (x2, $y_2$, $z_2$) are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

13. If a vector $\vec{r}$ has direction ratios proportional to a, b and c, then $\vec{r}=\dfrac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i}+b\hat{j}+c\hat{k})$.

14. Let $\vec{a}$ and $\vec{b}$ be any two vectors and k and m be two scalars, then
    (i) k$\vec{a}$ + m$\vec{a}$ = (k + m) $\vec{a}$
    (ii) k(m $\vec{a}$ ) = (km) $\vec{a}$
    (iii) k($\vec{a}$ + $\vec{b}$ ) = k $\vec{a}$ + k $\vec{b}$

15. Vectors $\vec{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r_2}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ are collinear if

    $(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})=k(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})$

    i.e. $x_1 = kx_2$; $y_1 = ky_2$; $z_1 = kz_2$

    or $\dfrac{x_1}{x_2}=\dfrac{y_1}{y_2}=\dfrac{z_1}{z_2}=k$

16. The projections of a vector $\vec{r}$ = x$\hat{i}$ + y$\hat{j}$ + z$\hat{k}$ on the coordinate axes are l$\vec{r}$, m$\vec{r}$, n$\vec{r}$, where I, m and n are the direction cosines of the vector $\vec{r}$.

17. The scalar product of vectors a and b is $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, where $\theta$ is the angie between vectors $\vec{a}$ and $\vec{b}$.

18. The scalar product in terms of components: Let $\vec{a}$ and $\vec{b}$ be two vectors such that

    $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

19. The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$ or $\theta=\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right)$.

20. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} - b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then the angle between the two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\theta = \cos^{-1}\left\{\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}\right\}$$

21. Let $\vec{a}$ and $\vec{b}$ be two vectors. Then
    (i) $\vec{a} . \vec{b} = 0$ if and only if $\vec{a}$ is perpendicular to $\vec{b}$.
    (ii) $\vec{a} . \vec{b} > 0$ if and only if $\theta$ is acute.
    (iii) $\vec{a} . \vec{b} < 0$ if and only if $\theta$ is obtuse.

22. Components of a vector along and perpendicular to vector: Let $\vec{a}$ and $\vec{b}$ be two vectors. Then the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ respectively.

23. The projection of $\vec{r}$ on the X, Y and Z axes are x, y and z, respectively, where $\vec{r} - x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

24. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then x, y and z are called the components of $\vec{r}$ along X, Y and Z axes, respectively.

25. If $\vec{r} - x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is a vector making an angle $\alpha$, $\beta$ and $\gamma$ with the X, Y and Z axes, respectively, then, $\vec{r} = |\vec{r}|\left[(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}\right]$ *and* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

26. For unit vectors $\hat{\imath}$ , $\hat{\jmath}$ and $\hat{k}$ , $\hat{\imath} . \hat{\imath} = \hat{\jmath} . \hat{\jmath} = \hat{k} . \hat{k} = 1$ and $\hat{\imath} . \hat{\jmath} = \hat{\jmath} . \hat{k} = \hat{k} . \hat{\imath} = 0$

27. The unit vector in the direction of vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

28. Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is given by:

$$\vec{a}\cdot\vec{b} = \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right) = \frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$$

29. Cauchy-Schwarz Inequality:

$$|\vec{a}\cdot\vec{b}| \le |\vec{a}|\cdot|\vec{b}|$$

30. Triangle Inequality:

$$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$$

31. The vector product of vectors a and b is $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}.$

32. Geometrical meaning of vector product:
    Let $\vec{a}$ and b be two non-zero, non-parallel vectors.
    Then the cross-product $\vec{a} \times \vec{b}$ is a vector with magnitude equal to the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides and direction $\hat{n}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

**Class : 12th Maths**
**Chapter- 10 : Vector Algebra**

**Vector Algebra**

**Vectors**

A quantity that has both magnitude and direction is called a vector. The distance between the initial and terminal points of a vector is called its magnitude. Magnitude of vector $\overrightarrow{AB}$ is $|AB|$.

**Multiplication of vector by a scalar**

If λ Multiplied to vector AB, then the magnitude is multiplied |λ| by and direction remain same (or opp.) according as λ is the + *ve* or, *ve*-

**Unit Vectors**

For a given vector $a$, the vector $\hat{a} = \frac{a}{|a|}$ gives the unit vector in the direction of $a$. for eg , if $a = 5i$, then $\hat{a} = \frac{5i}{|5|} = i$ ,which is a unit vector.

**Position Vectors**

Position vector of a point $P\ (x, y, z)$ is $x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude is $OP(r) = \sqrt{x^2 + y^2 + z^2}$ .For eg: Position vector of $P\ (2,3,5)$ is $2\hat{i} + 3\hat{j} + 5\hat{k}$ and its magnitude is $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$.

**Position of vectors**

The Position vector of a point R dividing a line segment joining $P,Q$ whose position vectors are $a, b$ resp., in the ratio $m : n$
(i) internally is $\frac{na + mb}{m + n}$, (ii) externally is $\frac{mb - na}{m - n}$

**Scalar Product of two vectors**

If a, b are the vectors and 'θ', angle between them, then their scalar product $a.b = |a||b|\cos\theta$
$\Rightarrow \cos\theta = \frac{a.b}{|a||b|}$

**Cross Product of two vectors**

$a \times b = |a||b|\sin\theta\,\hat{n}$, $\hat{n}$ is a unit vector perpendicular to line joining $a,b$.

**Direction ration and direction cosines**

The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
The magnitude $(r)$ direction ratios $(a,b,c)$ and direction cosines $(l,m,n)$ of vector $a\hat{i} + b\hat{j} + c\hat{k}$ are related as:
$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$
For eg : If $\overrightarrow{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$, then $r = \sqrt{1+4+9} = \sqrt{14}$
Direction ratios are (1,2,3) and direction cosines are $\left(\sqrt{\frac{1}{14}}, \sqrt{\frac{2}{14}}, \sqrt{\frac{3}{14}}\right)$

**Types of Vectors**

(i) Zero vector (initial and terminal points coincide)
(ii) Unit vector (magnitude is unity.
(iii) Coinitial vectors (same initial points)
(iv) Collinear vectors (parellel to the same line)
(v) Equal vectors (same magnitude and direction)
(vi) Negative of a vector (same magnitude, opp. direction)

**Properties of Vectors**

If we have two vectors $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ is any scalar, then-

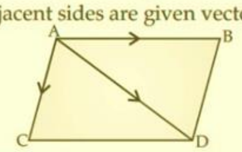
$a + b = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

$\lambda a = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

$a.b = a_1 b_1 + a_2 b_2 + a_3 b_3$ and

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

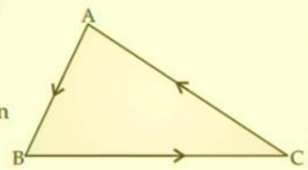
The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.




if $\overline{AB}, \overline{AC}$ are the given vectors, then $\overline{AB} + \overline{AC} = \overline{AD}$

The vector sum of the three sides of a triangle taken in order is 0 .i.e

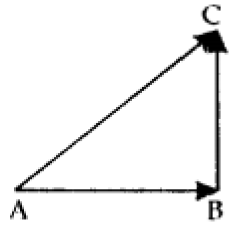
if $ABC$ is given triangle, then $\overline{AB} + \overline{BC} + \overline{CA} = 0$.



# Important Questions

## Multiple Choice questions-

1. In ΔABC, which of the following is not true?



(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

(d) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

2. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(a) $\vec{b} = \lambda\vec{a}$ tor some scalar λ.

(b) $\vec{a} = \pm\vec{b}$

(c) the respective components of $\vec{a}$ and $\vec{b}$ are proportional

(d) both the vectors $\vec{a}$ and $\vec{b}$ have the same direction, but different magnitudes.

3. If a is a non-zero vector of magnitude 'a' and λa non-zero scalar, then $\lambda\vec{a}$ is unit vector if:

(a) $\lambda = 1$

(b) $\lambda = -1$

(c) $a = |\lambda|$

(d) $a = \frac{1}{|\lambda|}$

4. Let λ be any non-zero scalar. Then for what possible values of x, y and z given below, the vectors $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $x\hat{\imath} - y\hat{\jmath} + z\hat{k}$ are perpendicular:

(a) $x = 2\lambda.\ y = \lambda, z = \lambda$

(b) $x = \lambda, y = 2\lambda, z = -\lambda$

(c) $x = -\lambda, y = 2\lambda, z = \lambda$

(d) $x = -\lambda, y = -2\lambda, z = \lambda.$

5. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is:

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$

6. Area of a rectangle having vertices

A $(-\hat{\imath} + \frac{1}{2}\hat{\jmath} + 4\hat{k})$,

B $(\hat{\imath} + \frac{1}{2}\hat{\jmath} + 4\hat{k})$,

C $(\hat{\imath} - \frac{1}{2}\hat{\jmath} + 4\hat{k})$,

D $(-\hat{\imath} - \frac{1}{2}\hat{\jmath} + 4\hat{k})$ is

(a) $\frac{1}{2}$ square unit

(b) 1 square unit

(c) 2 square units

(d) 4 square units.

7. If θ is the angle between two vectors $\vec{a}, \vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when

(a) $0 < \theta < \frac{\pi}{2}$

(b) $0 \leq \theta \leq \frac{\pi}{2}$

(c) $0 < \theta < \pi$

(d) $0 \leq \theta \leq \pi$

8. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and 6 is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if:

(a) $\theta = \frac{\pi}{4}$

(b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{2}$

(d) $\theta = \frac{2\pi}{3}$

9. If $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are the usual three perpendicular unit vectors, then the value of:

$\hat{\imath}.(\hat{\jmath} \times \hat{k}) + \hat{\jmath}.(\hat{\imath} \times \hat{k}) + \hat{k}.(\hat{\imath} \times \hat{\jmath})$ is

(a) 0

(b) -1

(c) 1

(d) 3

10. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to:

(a) 0

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π

## Very Short Questions:

1. Classify the following measures as scalar and vector quantities:

(i) 40°

(ii) 50 watt

(iii) $10 gm/cm^3$

(iv) 20 m/sec towards north

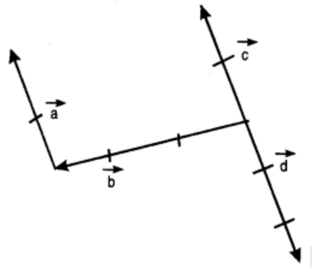
(v) 5 seconds.

2. In the figure, which of the vectors are:

(i) Collinear

(ii) Equal

(iii) Co-initial.




3. Find the sum of the vectors:

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \vec{a} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} + 6\hat{j} - 7\hat{k}$

4. Find the vector joining the points P (2,3,0) and Q (-1, – 2, – 4) directed from P to Q.

5. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of x + y + z.

6. Find the unit vector in the direction of the sum of the vectors:

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$

7. Find the value of 'p' for which the vectors: $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.

8. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$

9. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$

10. Find the area of the parallelogram whose diagonals are represented by the vectors: $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

## Short Questions:

1. If θ is the angle between two vectors:

$\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find sin θ.

2. X and Y are two points with position vectors $\overrightarrow{3a} + \vec{b}$ and $\vec{a} - \overrightarrow{3b}$ respectively. Write the po-sition vector of a point Z which divides the line segment XY in the ratio 2:1 externally.

3. Find the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, where:

$\vec{a} = 4\hat{i} - \hat{j} + 8\hat{k}$ and $\vec{b} = -\hat{j} + \hat{k}$

4. If $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

5. Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ be two vectors. Show that the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular to each other.

6. If the sum of two-unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

7. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{a}| = 7$, then find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$

8. Find $|\vec{a} - \vec{b}|$, if two vectors a and b are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$.

## Long Questions:

1. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a} = 21$.

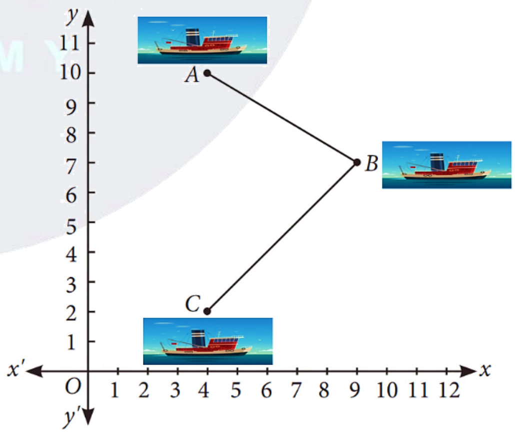
2. If $\vec{p} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{q} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude $5\sqrt{3}$ units perpendicular to the vector $\vec{q}$. and coplanar with vector $\vec{p}$ and $\vec{q}$.

3. If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find

4. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

## Case Study Questions:

1. A barge is pulled into harbour by two tug boats as shown in the figure.




Based on the above information, answer the following questions.

i. Position vector of A is:

a. $4\hat{i} + 2\hat{j}$

b. $4\hat{i} + 10\hat{j}$

c. $4\hat{i} - 10\hat{j}$

d. $4\hat{i} - 2\hat{j}$

ii. Position vector of B is:

a. $4\hat{i}+4\hat{j}$

b. $6\hat{i}+6\hat{j}$

c. $9\hat{i}+7\hat{j}$

d. $3\hat{i}+3\hat{j}$

iii. Find the vector $\overline{AC}$ in terms of $\hat{i},\hat{j}$.

a. $8\hat{j}$

b. $-8\hat{j}$

c. $8\hat{i}$

d. None of these

iv. If $\vec{A}=\hat{i}+2\hat{j}=3\hat{k}$, then its unit vector is:

a. $\frac{\hat{i}}{\sqrt{14}}+\frac{2\hat{j}}{\sqrt{14}}+\frac{3\hat{k}}{\sqrt{14}}$

b. $\frac{3\hat{i}}{\sqrt{14}}+\frac{2\hat{j}}{\sqrt{14}}+\frac{\hat{k}}{\sqrt{14}}$

c. $\frac{2\hat{i}}{\sqrt{14}}+\frac{3\hat{j}}{\sqrt{14}}+\frac{\hat{k}}{\sqrt{14}}$

d. None of these

v. If $\vec{A}=4\hat{i}+3\hat{j}$ and $\vec{B}=3\hat{i}+4\hat{j}$, then $|\vec{A}|+|\vec{B}|=$

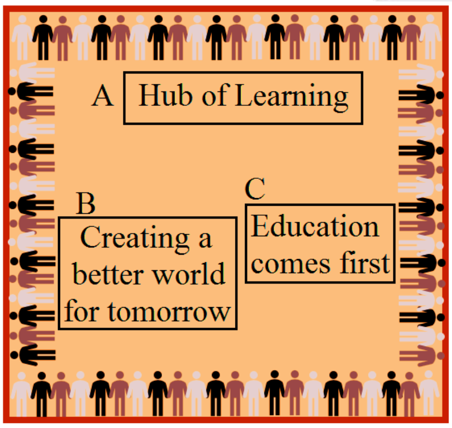
a. 12

b. 13

c. 14

d. 10

**2.** Three slogans on chart papers are to be placed on a school bulletin board at the points A, Band C displaying A (Hub of Learning), B (Creating a better world for tomorrow) and C (Education comes first). The coordinates of these points are (1, 4, 2), (3, -3, -2) and (-2, 2, 6) respectively.




Based on the above information, answer the following questions.

i. Let $\vec{a},\vec{b}$ and $\vec{c}$ be the position vectors of points A, B and C respectively, then $\vec{a}+\vec{b}+\vec{c}$ is equal to:

a. $2\hat{i}+3\hat{j}+6\hat{k}$

b. $2\hat{i}-3\hat{j}-6\hat{k}$

c. $2\hat{i}+8\hat{j}+3\hat{k}$

d. $2(7\hat{i}+8\hat{j}+3\hat{k})$

ii. Which of the following is not true?

a. $\overline{AB}+\overline{BC}+\overline{CA}=\vec{0}$

b. $\overline{AB}+\overline{BC}-\overline{AC}=\vec{0}$

c. $\overline{AB}+\overline{BC}-\overline{CA}=\vec{0}$

d. $\overline{AB}-\overline{CB}+\overline{CA}=\vec{0}$

iii. Area of $\Delta$ABC is:

a. 19 sq. units

b. $\sqrt{1937}$ sq. units

c. $\frac{1}{2}\sqrt{1937}$ sq. units

d. $\sqrt{1837}$ sq. units

iv. Suppose, if the given slogans are to be placed on a straight line, then the value of $|\vec{a}\times\vec{b}+\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$ will be equal to:

a. $-1$

b. $-2$

c. 2

d. 0

v. If $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$, then unit vector in the direction of vector $\vec{a}$ is:

a. $\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}$

b. $\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$

c. $\frac{3}{7}\hat{i}+\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}$

d. None of these

# Answer Key

## Multiple Choice questions-

1. **Answer:** (c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$
2. **Answer:** (d) both the vectors $\vec{a}$ and $\vec{b}$ have the same direction, but different magnitudes.
3. **Answer:** (d) a = $\frac{1}{|\lambda|}$
4. **Answer:** (c) x = -λ, y = 2λ, z = λ
5. **Answer:** (b) $\frac{\pi}{4}$
6. **Answer:** (c) 2 square units
7. **Answer:** (b) $0 \le \theta \le \frac{\pi}{2}$
8. **Answer:** (d) $\theta = \frac{2\pi}{3}$
9. **Answer:** (d) 3
10. **Answer:** (b) $\frac{\pi}{4}$

## Very Short Answer:

**1. Solution:**

(i) Angle-scalar
(ii) Power-scalar
(iii) Density-scalar
(iv) Velocity-vector
(v) Time-scalar.

**2. Solution:**

(i) $\vec{a}$, $\vec{c}$ and $\vec{a}$ are collinear vectors.
(ii) $\vec{a}$ and $\vec{c}$ are equal vectors.
(iii) $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-initial vectors.

**3. Solution:**

Sum of the vectors = $\hat{a} + \hat{b} + \hat{c}$

$=(\hat{i}-2\hat{j}+\hat{k})+(-2\hat{i}+4\hat{j}+5\hat{k})+(\hat{i}-6\hat{j}-7\hat{k})$

$=(\hat{i}-2\hat{i}+\hat{i})+(-2\hat{i}+4\hat{j}-6\hat{j})+(\hat{k}+5\hat{k}-7\hat{k})$

$=-4\hat{j}-\hat{k}.$

**4. Solution:**

Since the vector is directed from P to Q,

∴ P is the initial point and Q is the terminal point.

∴ Reqd. vector = $\overrightarrow{PQ}$

$=(-\hat{i}-2\hat{j}-4\hat{k})-(2\hat{i}+3\hat{j}+0\hat{k})$

$=(-1-2)\hat{i}+(-2-3)\hat{j}+(-4-0)\hat{k}$

$=-3\hat{i}-5\hat{j}-4\hat{k}.$

**5. Solution:**

Here

$\vec{a}=\vec{b} \Rightarrow x\hat{i}+2\hat{j}-z\hat{k}=3\hat{i}-y\hat{j}+\hat{k}$

Comparing, A: = 3,2 = -y i.e. y = -2, ~z= 1 i.e. z = -1.

Hence, x + y + z = 3 – 2 – 1 = 0.

**6. Solution:**

We have : $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}+3\hat{k}$

$\therefore \vec{c}=\vec{a}+\vec{b}$

$=(2\hat{i}-\hat{j}+2\hat{k})+(-\hat{i}+\hat{j}+3\hat{k})$

$=\hat{i}+0\cdot\hat{j}+5\hat{k}$

$\therefore |\vec{c}|=\sqrt{1^2+0^2+5^2}=\sqrt{1+0+25}=\sqrt{26}$

$\therefore$ Reqd. unit vector $=\hat{c}=\frac{\vec{c}}{|\vec{c}|}$

$=\frac{\hat{i}+0\hat{j}+5\hat{k}}{\sqrt{26}}=\frac{\hat{i}+5\hat{k}}{\sqrt{26}}$

**7. Solution:**

The given vectors $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\hat{\imath} - 2p\hat{\jmath} + 3\hat{k}$ are parallel

If $\frac{3}{1}=\frac{2}{-2p}=\frac{9}{3}$ if $3=\frac{1}{-p}=3$

If $p=-\frac{1}{3}$

**8. Solution:**

We have : $|\vec{a}+\vec{b}|=13$

Squaring, $(\vec{a}+\vec{b})^2=169$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=169$

$\Rightarrow (5)^2+|\vec{b}|^2+2(0)=169$

$[\because \vec{a}$ and $\vec{b}$ are perpendicular $\Rightarrow \vec{a}\,\vec{b}=0]$

$\Rightarrow |\vec{b}|^2=169-25=144$

Hence, $|\vec{b}|=12$

**9. Solution:**

Be the question, $|\vec{a}|=|\vec{b}|$ ....(1)

Now $\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \frac{9}{2}=|\vec{a}|\,|\vec{b}|\cos 60°$ [Using (1)]

$\Rightarrow \frac{9}{2}=|\vec{a}|^2\left(\frac{1}{2}\right)$

$\Rightarrow |\vec{a}|^2=9.$

Hence, $|\vec{a}|=|\vec{b}|=3$

**10. Solution:**

We have : $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+2\hat{k}$

$$\therefore\ \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 4\\ 2 & -1 & 2\end{vmatrix}$$

$$=\hat{i}(-6+4)-\hat{j}(4-8)+\hat{k}(-2+6)$$

$$=-2\hat{i}+4\hat{j}+4\hat{k}$$

$$\therefore\ |\vec{a}\times\vec{b}|=\sqrt{4+16+16}=\sqrt{36}=6$$

$$\therefore\ \text{Area of the parallelogram}=\frac{1}{2}|\vec{a}\times\vec{b}|$$

$$=\frac{1}{2}(6)=3\ \text{sq. units.}$$

## Short Answer:

**1. Solution:**

We know that $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$$\Rightarrow\ \sin\theta=\frac{(\hat{i}-2\hat{j}+3\hat{k})\times(3\hat{i}-2\hat{j}+\hat{k})}{|\hat{i}-2\hat{j}+3\hat{k}||3\hat{i}-2\hat{j}+\hat{k}|}\quad ...(1)$$

Now $(\hat{i}-2\hat{j}+3\hat{k})\times(3\hat{i}-2\hat{j}+\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 3 & -2 & 1\end{vmatrix}$$

$$=\hat{i}(-2+6)-\hat{j}(1-9)+\hat{k}(-2+6)$$

$$=4\hat{i}+8\hat{j}+4\hat{k}$$

$$\therefore\ |4\hat{i}+8\hat{j}+4\hat{k}|=\sqrt{16+64+16}$$

$$=\sqrt{96}=4\sqrt{6}\ \text{and}\ |\hat{i}-2\hat{j}+3\hat{k}|=\sqrt{1+4+9}=\sqrt{14};$$

$$|3\hat{i}-2\hat{j}+\hat{k}|=\sqrt{9+4+1}=\sqrt{14}$$

$$\therefore\ \text{From (1)},\ \sin\theta=\frac{4\sqrt{6}}{\sqrt{14}\sqrt{14}}=\frac{4\sqrt{6}}{14}$$

Hence, $\sin\theta=\dfrac{2\sqrt{6}}{7}$.

**2. Solution:**

Position vector of

$$A=\frac{2(\vec{a}-3\vec{b})-(3\vec{a}+\vec{b})}{2-1}=\vec{a}-7\vec{b}$$

**3. Solution:**

We have: $\vec{a}=4\hat{i}-\hat{j}+8\hat{k},\ \vec{b}=-\hat{j}+\hat{k}$

$$\therefore\ \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 8\\ 0 & -1 & 1\end{vmatrix}$$

$$=\hat{i}(-1+8)-\hat{j}(4-0)+\hat{k}(-4+0)=7\hat{i}-4\hat{j}-4\hat{k}$$

$$\therefore\ |\vec{a}\times\vec{b}|=\sqrt{(7)^2+(-4)^2+(-4)^2}$$

$$=\sqrt{49+16+16}=\sqrt{81}=9$$

Hence, the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\frac{7\hat{i}-4\hat{j}-4\hat{k}}{9}=\frac{7}{9}\hat{i}-\frac{4}{9}\hat{j}-\frac{4}{9}\hat{k}.$$

**4. Solution:**

We have: $\vec{a}=2\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$

$$\therefore \vec{a}+\lambda\vec{b}=(2\hat{i}+2\hat{j}+3\hat{k})+\lambda(-\hat{i}+2\hat{j}+\hat{k})$$

$$=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$$

Now, $(\vec{a}+\lambda\vec{b})$ is perpendicular to c,

$$\therefore (\vec{a}+\lambda\vec{b})\cdot\vec{c}=0$$

$$\Rightarrow ((2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda\hat{k})\cdot(3\hat{i}+\hat{j}))=0$$

$\Rightarrow (2 - \lambda)(3) + (2 + 2\lambda)(1) + (3 + \lambda)(0) = 0$

$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$

$\Rightarrow -\lambda, + 8 = 0.$

Hence, $\lambda, = 8.$

**5. Solution:**

Here, $\vec{a}+\vec{b}=(\hat{i}+2\hat{j}-3\hat{k})+(3\hat{i}-\hat{j}+2\hat{k})=4\hat{i}+\hat{j}-\hat{k}$

and

$$\vec{a}-\vec{b}=(\hat{i}+2\hat{j}-3\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=-2\hat{i}+3\hat{j}-5\hat{k}$$

Now, $\vec{a}+\vec{b}\cdot\vec{a}-\vec{b}=(4\hat{i}+\hat{j}-\hat{k})-(-2\hat{i}+3\hat{j}-5\hat{k})$

$= (4)(-2) + (1)(3) + (-1)(-5)$

$= -8 + 3 + 5 = 0.$

Hence $\vec{a}+\vec{b}$ is perpendicular to $\vec{a}-\vec{b}$.

**6. Solution:**

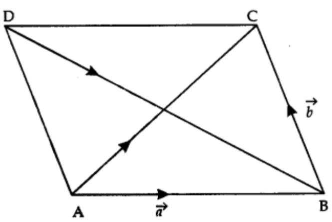
We have: $|\vec{a}|=|\vec{b}|=1, |\vec{a}+\vec{b}|=1.$

Let $\vec{AB}=\vec{a}, \vec{BC}=\vec{b}$

Then, $\vec{AC}=\vec{AB}+\vec{BC}=\vec{a}+\vec{b}$

and $\vec{DB}=\vec{DA}+\vec{AB}=-\vec{AD}+\vec{AB}=\vec{AB}-\vec{AD}$

$$=\vec{a}-\vec{b}$$



By the question,

$|\vec{AB}| = |\vec{BC}| = |\vec{AC}| = 1$

⇒ ΔABC is equilateral, each of its angles being 60°

⇒ ∠DAB = 2 x 60° = 120° and ∠ADB = 30°.

By Sine Formula,

$$\frac{DB}{\sin\angle DAB} = \frac{AB}{\sin\angle ADB}$$

$$\Rightarrow \frac{|\vec{DB}|}{\sin 120^\circ} = \frac{|\vec{AB}|}{\sin 30^\circ}$$

$$\Rightarrow |\vec{DB}| = \frac{\sin 120^\circ}{\sin 30^\circ}|\vec{AB}|$$

$$= \frac{\sqrt{3}/2}{1/2} \times 1 = \sqrt{3}$$

Hence, $|\vec{a} - \vec{b}| = \sqrt{3}.$

**7. Solution:**

Here, $\vec{a} + \vec{b} + \vec{c} = 0$

$$\Rightarrow (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c} = 0$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow 3^2+5^2+7^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = -(9+25+49)$$

Hence, $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -\frac{83}{2}.$

**8. Solution:**

Here, $\vec{a} + \vec{b} + \vec{c} = 0$

$$\Rightarrow (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c} = 0$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow 3^2+5^2+7^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = -(9+25+49)$$

Hence, $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -\frac{83}{2}.$

## Long Answer:

**1. Solution:**

We have: $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$

$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

Since $\vec{d}$ is perpendicular to both $\vec{c}$ and $\vec{b}$

$\vec{d}\cdot\vec{c} = 0$ and $\vec{d}\cdot\vec{b} = 0$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}+\hat{j}-\hat{k}) = 0$$

and $(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-4\hat{j}+5\hat{j}) = 0$

⇒ 3x + y - z = 0 ...(1)

and x - 4y + 5z = 0 ...(2)

Also, $\vec{d}\cdot\vec{a} = 21$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(4\hat{i}+5\hat{j}-\hat{k}) = 21$$

⇒ 4x + 5y-z = 21 ...(3)

Multiplying (1) by 5,

1 5x + 5y – 5z = 0 ...(4)

Adding (2) and (4),

16x + y = 0 ...(5)

Subtracting (1) from (3),

x + 4y = 21 ...(6)

From (5),

y = -16x ...(7)

Putting in (6),

x – 64x = 21

-63x: = 21

Putting in (7), $y = -16\left(-\frac{1}{3}\right) = \frac{16}{3}$

Putting in (1), $3\left(-\frac{1}{3}\right) + \frac{16}{3} - z = 0$

z = 13/3

Hence, $\vec{d} = -\frac{1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$

**2. Solution:**

Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be the vector.

Since $\vec{r} \perp \vec{q}$

(1) (a) + (-2) (b) + 1 (c) = 0

⇒ a – 2b + c = 0

Again, $\vec{p}, \vec{q}$ and $\vec{r}$ and coplanar,

$\therefore [\vec{p}\,\vec{q}\,\vec{r}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ a & b & c \end{vmatrix} = 0$$

⇒ (1) (-2c – b) – (1) (c – a) + (1) (b + 2a) = 0

$\Rightarrow$ -2c-b-c + a + b + 2a = 0

$\Rightarrow$ 3a – 3c = 0

$\Rightarrow$ a – c – 0

Solving (1) and (2),

$$\frac{a}{2-0}=\frac{b}{1+1}=\frac{c}{0+2}$$

$$\Rightarrow \frac{a}{2}=\frac{b}{2}=\frac{c}{2}$$

$$\Rightarrow \frac{a}{1}=\frac{p}{1}=\frac{c}{1}$$

$$\therefore \vec{r}=1\hat{i}+1\hat{j}+1\hat{k}$$

$$\therefore |\vec{r}|=\sqrt{3}$$

$$\therefore \text{ Unit vector } \hat{r}=\frac{\vec{r}}{|\vec{r}|}=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

Hence, the required vector $=5\sqrt{3}\,\hat{r}$

$$=5\sqrt{3}\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)=5(\hat{i}+\hat{j}+\hat{k}).$$

**3. Solution:**

Note: If 'θ' is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$=(2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})=\hat{i}+4\hat{j}-\hat{k}$$

$$\therefore |\overrightarrow{AB}|=\sqrt{(1)^2+(4)^2+(-1)^2}=3\sqrt{2}$$

Similarly, $\overrightarrow{CD}=-2\hat{i}-8\hat{j}+2\hat{k}$ and $|\overrightarrow{CD}|=6\sqrt{2}.$

$$\text{Thus } \cos\theta=\frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$$

$$=\frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})}=\frac{-36}{36}=-1$$

Since $0\le\theta\le\pi$, it follows that $\theta=\pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Alternatively, $\overrightarrow{AB}=-\frac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

**4. Solution:**

Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\therefore \vec{a}+\vec{b}=-\vec{c}$

Squaring, $(\vec{a}+\vec{b})=\vec{c}^2$

$$\Rightarrow \vec{a}^2+\vec{b}^2+2\vec{a}\cdot\vec{b}=\vec{c}^2$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta=|\vec{c}|^2$$

where 'θ' is the angle between a and b

$\Rightarrow$ (3)2 + (5)2 + 2 (3) (5) cos θ = (7)2

$\Rightarrow$9 + 25 + 30 cos θ = 49

$\Rightarrow$ 30 cos θ = 49 – 34 $\Rightarrow$ cos θ = $\frac{1}{2}$

$\Rightarrow$ θ = 60°.

Hence, the angle between $\vec{a}$ and $\vec{b}$ is 60°.

## Case Study Answers:

**1. Answer :**

i. (b) $4\hat{i}+10\hat{j}$

**Solution:**

Here, (4, 10) are the coordinates of A.

$\therefore$ P.V. of A $=4\hat{i}+10\hat{j}$

ii. (c) $9\hat{i}+7\hat{j}$

**Solution:**

Here, (9, 7) are the coordinates of B.

$\therefore$ P.V. of A $=9\hat{i}+7\hat{j}$

iii. (b) $-8\hat{j}$

**Solution:**

Here, P.V. of A $=4\hat{i}+10\hat{j}$ and P.V. of C $=4\hat{i}+2\hat{j}$

$$\therefore \overrightarrow{AC}=(4-4)\hat{i}+(2-10)\hat{j}=-8\hat{j}$$

iv. (a) $\frac{\hat{i}}{\sqrt{14}}+\frac{2\hat{j}}{\sqrt{14}}+\frac{3\hat{k}}{\sqrt{14}}$

**Solution:**

Here, $\vec{A}=\hat{i}+2\hat{j}+3\hat{k}$

$$\therefore |\vec{A}|=\sqrt{1^2+2^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$\therefore \hat{A}=\frac{\vec{A}}{|\vec{A}|}=\frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$$

$$=\frac{1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}$$

v. (d) 10

**Solution:**

We have, $\vec{A}=4\hat{i}+3\hat{j}$ and $\vec{B}=3\hat{i}+4\hat{j}$

$$\therefore |\vec{A}|=\sqrt{4^2+3^2}=\sqrt{16+9}=\sqrt{25}=5$$

$$\text{and } |\vec{B}|=\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5$$

Thus, $|\vec{A}|+|\vec{B}|=5+5=10.$

**2. Answer:**

i. (a) $2\hat{i}+3\hat{j}+6\hat{k}$

**Solution:**

$\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-3\hat{j}-2\hat{k}$

and $\vec{c}=2\hat{i}+2\hat{j}+6\hat{k}$

$\therefore\ \vec{a}+\vec{b}+\vec{c}=2\hat{i}+3\hat{j}+6\hat{k}$

ii. (c) $\overline{AB}+\overline{BC}-\overline{CA}=\vec{0}$

**Solution:**

Using triangle law of addition in ΔABC,

We get $\overline{AB}+\overline{BC}-\overline{CA}=\vec{0}$ which can be rewritten as,

$\overline{AB}+\overline{BC}-\overline{CA}=\vec{0}\ or\ \overline{AB}-\overline{CB}+\overline{CA}=\vec{0}$

iii. (c) $\frac{1}{2}\sqrt{1937}$ sq. units

**Solution:**

We have, A(1, 4, 2), B(3, –3, 2) and C(–2, 2, 6)

Now, $\overline{AB}=\vec{b}-\vec{a}=2\hat{i}-7\hat{j}-4\hat{k}$

and $\overline{AC}=\vec{c}-\vec{a}=-3\hat{i}-2\hat{j}+4\hat{k}$

$$\therefore\ \overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -7 & -4\\ -3 & -2 & 4\end{vmatrix}$$

$=\hat{i}(-28-8)-\hat{j}(8-12)+\hat{k}(-4-21)$

$=-36\hat{i}+4\hat{j}-25\hat{k}$

Now, $\left|\overline{AB}\times\overline{AC}\right|=\sqrt{(-36)^2+4^2+(-25)^2}$

$=\sqrt{1296+16+625}=\sqrt{1937}$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$

$=\frac{1}{2}\sqrt{1937}$ sq. units

iv. (d) 0

**Solution:**

If the given points lie on the straight line, then the points will be collinear and so area of ΔABC = 0

$\Rightarrow \left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|=0$

[$\because$ If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the three vertices A, B and C of ΔABC, then area of triangle $=\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$]

v. (b) $\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$

**Solution:**

Here,

$|\vec{a}|=\sqrt{2^2+3^2+6^2}=\sqrt{4+6+36}=\sqrt{49}=7$

$\therefore$ Unit vector in the direction of vector $\vec{a}$ is

$\hat{a}=\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$

# Three Dimensional Geometry 11

## Top Review

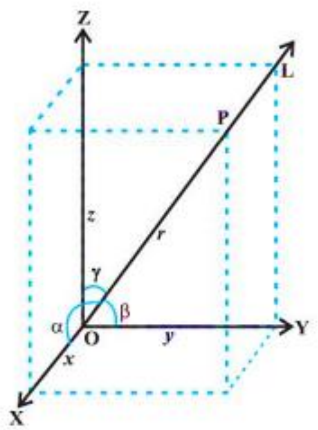
1. If P($x_1$, $y_1$, $z_1$) and Q($x_2$, $y_2$, $z_2$) are two points in space, then $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
2. The distance of a point P($x_1$, $y_1$, $z_1$) from the origin 0 is given by $OP=\sqrt{x_1^2+y_1^2+z_1^2}$
3. The coordinates of a point R which divides the line joining the points P($x_1$, $y_1$, $z_1$) and Q($x_2$, $y_2$, $z_2$ ) internally in the ratio m : n are $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$.
4. The coordinates of a point R which divides the line joining the points P($x_1$, $y_1$, $z_1$) and Q($x_2$ $y_2$, $z_2$) externally in the ratio m : n are $\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n},\frac{mz_2-nz_1}{m-n}\right)$.
5. Let P($x_1$, $y_1$ $z_1$) and Q($x_2$ $y_2$, $z_2$) be two points in space. The coordinates of the midpoint of PQ are $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$.
6. Let P($x_1$, $y_1$ $z_1$), Q($x_2$ $y_2$, $z_2$) and R($x_3$ $y_3$, $z_3$) be three vertices of the triangle.

   $G(x,y,z)$ is $G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$.

   Hence, the centroid
7. The projection of the line joining points P($x_1$, $y_1$, $z_1$) and Q($x_2$, $y_2$, $z_2$) to the line with direction cosines, I, m and n is $|(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n|$.

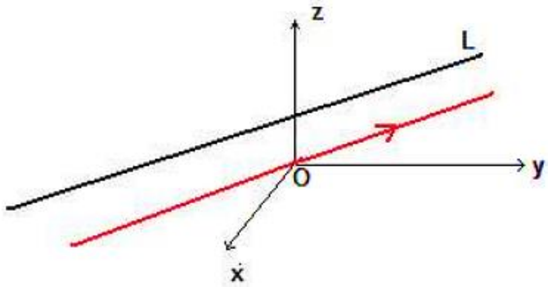
## Top Concepts

1. The angles $\alpha$, $\beta$ and $\gamma$ which a directed line L, through the origin, makes with the x , y and z axes, respectively, are called direction angles.




If the direction of line L is reversed, then the direction angles will $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$.

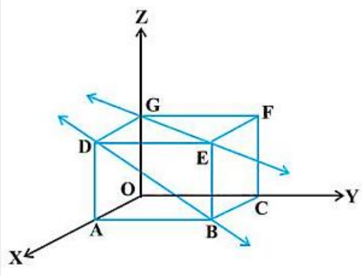
2. If a directed line L passes through the origin and makes angles $\alpha$ , $\beta$ and $\gamma$ with the x, y and z axes respectively, then
   $\lambda = \cos \alpha$ , m = $\cos \beta$ and n = $\cos \gamma$ are called direction cosines of line L.
3. For a given line to have a unique set of direction cosines, a directed line is used.
4. The direction cosines of the directed line which does not pass through the origin can be obtained by drawing a line parallel to it and passing through the origin.



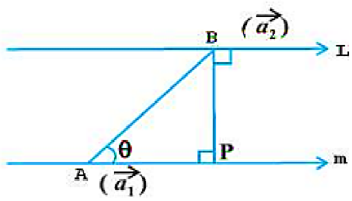

5. Any three numbers which are proportional to the direction cosines of the line are called direction ratios. If λ, m and n are the direction cosines and a, b and c are the direction ratios of a line, then λ = ka, m = kb and n = kc, where k is any non-zero real number.
6. For any line, there are an infinite number of direction ratios.
7. Direction ratios of the line joining P($x_1$, $y_1$, $z_1$) and Q($x_2$, $y_2$, $z_2$) may be taken as
   $x_2 - x_1, y_2 - y_1, z_2 - z_1$ or $x_1 - x_2, y_1 - y_2, z_1 - z_2$
8. Direction cosines of the x-axis are cos0, cos90, cos90, i.e., 1, 0, 0.
   Similarly, the direction cosines of the y-axis are 0, 1, 0 and the z-axis are 0, 0, 1, respectively.
9. A line is uniquely determined if
   1. It passes through a given point and has given direction ratios

   OR

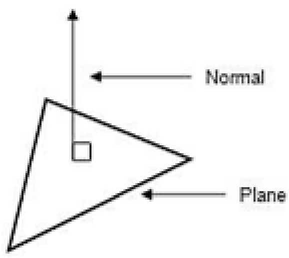
   2. It passes through two given points.
10. Two lines with direction ratios $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$, respectively, are perpendicular if
    $a_1b_1 + a_2b_2 + a_3b_3 = 0$
11. Two lines with direction ratios $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$, respectively, are parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$
12. The lines which are neither intersecting nor parallel are called as skew lines. Skew lines are non- coplanar, i.e., they do not belong to the same 2D plane.
    GE and DB are skew lines.



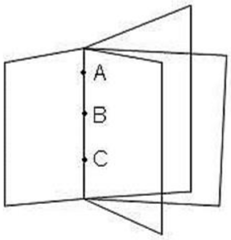

13. The **angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
14. If two lines in space are intersecting, then the shortest distance between them is zero.
15. If two lines in space are parallel, then the shortest distance between them is the perpendicular distance.




16. The normal vector, often simply called the 'normal' to a surface, is a vector perpendicular to a surface.

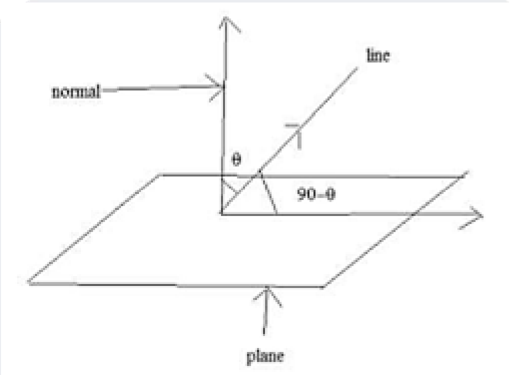


17. If the three points are collinear, then the line containing those three points can be part of many planes.

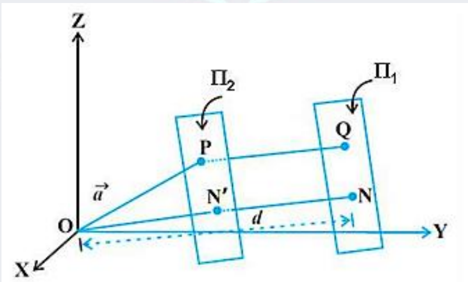



18. The angle between two planes is defined as the angle between their normals.
19. If the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular to each other, then $A_1A_2 + B_1B_2 + C_1C_2 = 0$

    If the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$
20. The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.




21. The distance of a point from a plane is the length of the unique line from the point to the plane which is perpendicular to the plane.




## Top Formulae

1. Direction cosines of the line L are connected by the relation $l^2 + m^2 + n^2 = 1$.
2. If a, b, and c are the direction ratios of a line, and l, m, and n are its direction cosines, then

$$\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

3. The direction cosines of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$ where

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

4. Vector equation of a line which passes through the given point whose position vector is ä and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$
5. If coordinates of point A are $(x_1, y_1, z_1)$ and direction ratios of the line are a, b, c, then cartesian form of equation of line is: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

6. If coordinates of point A are $(x_1, y_1, z_1)$ and direction cosines of the line are l, m, and n, then Cartesian equation of line is: $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$
7. The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$
8. Cartesian equation of a line which passes through two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$
9. The parametric equations of the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ are $x = x_1 + ar, y = y_1 + br, z = z_1 = cr$, where $r \in R$
10. Equation of the x-axis: $\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{0}$ or $y = 0$ and $z = 0$
11. Equation of the y-axis: $\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{0}$ or $x = 0$ and $z = 0$
12. Equation of the z-axis: $\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{0}$ or $x = 0$ and $y = 0$
13. Conversion of a Cartesian form of an equation of a line to a vector form:

    Let the Cartesian form of an equation of line be $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

    Hence, the vector form of the equation of the line is $\vec{r}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$, where $\lambda$ is a parameter.
14. Conversion of a vector form of the equation of a line to the Cartesian form:

    Let the Cartesian form of the equation of a line be $\vec{r}=\vec{a}+\lambda\vec{m}$, where $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ and $\lambda$ is a parameter

    Then the Cartesian form of the equation of the line is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$.
15. Angle $\theta$ between two lines $L_1$ and $L_2$ passing through the origin and having direction ratios $a_1, b_1, C_1$ and $a_2, b_2, C_2$ is

    $$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

    Or $$\sin\theta=\frac{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2+c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$
16. Condition of perpendicularity: If the lines are perpendicular to each other, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
17. Condition of parallelism: If the lines are parallel, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
18. Equation of a line passing through a point having position vector $\vec{k}$ and perpendicular to the lines $\vec{r}=\vec{a_1}+m\vec{b_1}$ and $\vec{r}=\vec{a_2}+m\vec{b_2}$ is $\vec{k}=m(\overline{b_1}\times\overline{b_2})$
19. To find the intersection of two lines:

    Consider the two lines:

    $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

    **Step (i):** The general coordinates of general points on the given two lines are
    $(a_1k + x_1, b_1k + y_1, c_1k + z_1)$ and $(a_2m + x_2, b_2m + y_2, c_2m + z_2)$

**Step (ii):** Equate both the points

Thus, we have $a_1k + x_1 = a_2m + x_2$, $b_1k + y_1 = b_2m + y_2$, $C_1k + z_1 = c_2m + z_2$

**Step (iii):** Solve the first two equations to get the values of k and m. Check whether the point satisfies the third equation also. If it satisfies, then the lines intersect, otherwise they do not.

**Step (iv):** Substitute the values of k and m in the set of three equations to get the intersection point.

20. To find the intersection of two lines in the vector form:

Let the two lines be:

$$\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + k(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad ....(1)$$

$$\vec{r} = (a_1'\hat{i} + a_2'\hat{j} + a_3'\hat{k}) + m(b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}) \quad ....(2)$$

**Step (i):** Position vectors of arbitrary points on (1) and (2) are

$$(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + k(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$(a_1'\hat{i} + a_2'\hat{j} + a_3'\hat{k}) + m(b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k})$$

**Step (ii):** Because the lines (1) and (2) intersect, they intersect each other, and their points of intersection are as follows:

$a_1 + ka_1' = b_1 + mb_1'$: $a_2 + ka_2' = b_2 + mb_2'$; $a_3 + ka_3' = b_3 + mb_3'$

**Step (iii):** Solve any two of the equations to get the values of k and m. Substitute the values of k and m in the third equation to check whether it satisfies it. If it does satisfy it, then the two lines intersect, otherwise they do not.

**Step (iv):** Substitute the values of k and m to get the point of intersection.

21. Perpendicular distance of a line from a point: Let $P(u, v, w)$ be the given point.

Let $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ be the given line.

Let N be the foot of the perpendicular.

Then the coordinates of N are,

$$(x_1 + ak, y_1 + bk, z_1 + ck), where\ k = -\frac{a(u - x_1) + b(v - y_1) + c(w + z_1)}{a^2 + b^2 + c^2}$$

Now, the distance PN can be determined using the distance formula.

22. Perpendicular distance of a line from a point when it is in the vector form:

**Step (i):** Let $P(\vec{u})$ be the given point. Let $\vec{r}$ - $\vec{a}$ + k$\vec{b}$ be the position vector of the line.

**Step (ii):** Find $\overrightarrow{PN}$ - Position vector of N — Position vector of P

**Step (iii):** $\overrightarrow{PN}.\vec{b} = 0$

**Step (iv):** Get the value of k

**Step (v):** Substitute the value of k in $\vec{r}$ - $\vec{a}$ + k$\vec{b}$

**Step (vi):** Compute $|\overrightarrow{PN}|$ to obtain the perpendicular distance.

23. **Skew lines:** Two lines are said to be skew lines if they are neither parallel nor intersecting.

24. **Shortest distance:** The shortest distance between two lines $L_1$ and $L_2$ is the distance PQ between the points P and Q, where the lines of shortest distance intersect the two given lines.

25. The shortest distance between two skew lines L and M having equations

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ respectively, is

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2})\cdot(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

26. Condition for two given lines to intersect: If the lines $\vec{r} = \vec{a}_1 + k\vec{b}_1$ and $\vec{r} = \vec{a}_2 + k\vec{b}_2$ intersect, then the shortest distance between them is zero.

Thus, $\left|\frac{(\vec{b_1} \times \vec{b_2})\cdot(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right| = 0 \Rightarrow (\vec{b_1} \times \vec{b_2})\cdot(\vec{a_2} - \vec{a_1}) = 0$

27. The shortest distance between the lines in the Cartesian form

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

28. Distance between parallel lines $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$ is $d=\left|\frac{\vec{b}\times(\vec{a_2}-\vec{a_1})}{|\vec{b}|}\right|$.

29. The equation of a plane at a distance d from the origin where ñ is the unit vector normal to the plane, through the origin in vector form, is $\vec{r}\cdot\hat{n}=d$.

30. Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane are I, m, n is lx + my + nz = d.

31. The general equation of the plane is ax + by + cz + d = 0.

32. The equation of a plane perpendicular to a given vector $\vec{N}$ and passing through a given point $\vec{a}$ is $(\vec{r}-\vec{a})\cdot\vec{N}=0$.

33. The equation of a plane perpendicular to a given line with direction ratios A, B and C and passing through a given point $(x_1, y_1, z_1)$ is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

34. The equation of a plane passing through three non-collinear points in the vector form is given as $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$

35. Reduction of the vector form of the equation of a plane to the Cartesian equation:

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}, \vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$

Then the Cartesian equation of a plane is, $(x-a_1)n_1+(y-a_2)n_2+(z-a_3)n_3=0$

36. The vector equation of the plane passing through the points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

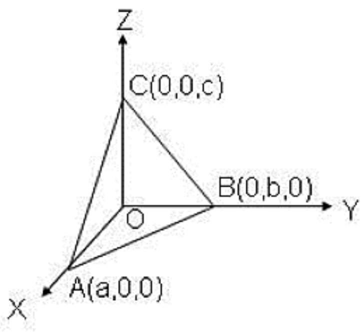
$\vec{r}=(1-m-n)\vec{a}+m\vec{b}+n\vec{c}$ (parametric form)

$\vec{r}\cdot(\vec{a}\times\vec{b})+\vec{r}\cdot(\vec{b}\times\vec{c})+\vec{r}\cdot(\vec{c}\times\vec{a})=\vec{a}\cdot(\vec{b}\times\vec{c})$ (non-parametric form)

37. The equation of a plane passing through three non-collinear points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ in the Cartesian form is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

38. The intercept form of the equation of a plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where a, b and c are the intercepts on the x, y and z-axes, respectively.



39. Any plane passing through the intersection of two planes $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$ is given by $\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$
40. The Cartesian equation of a plane passing through the intersection of two planes $A_1x+B_1y+C_1z=d_1$ and $A_2x+B_2y+C_2z=d_2$ is $(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0$
41. The equation of the planes bisecting the angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given by $\dfrac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$
42. The angle $\theta$ between a line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$ is given by the following relation:
$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\cdot\sqrt{l^2+m^2+n^2}}$$
43. If a line is perpendicular to a normal to the plane, then it is parallel to the plane.
44. If a line is parallel to a normal to the plane, then it is perpendicular to the plane.
45. The line $\vec{r}$ - $\vec{a}$ + k$\vec{b}$ lies in the plane $\vec{r}\,.\vec{n}$ = d if $\vec{a}\,.\,\vec{n}$ = d and $\vec{b}\,.\,\vec{n}$ = 0
46. The line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ lies in the plane $ax + by + cz + d = 0$ if $ax_1+by_1+cz_1+d=0$ and $al+bm+cn=0$.
47. The equation of a plane containing the line $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, where $al + bm + cn = 0$.
48. The given lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar if and only $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0$
49. Let $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ be the coordinates of the points M and N, respectively. Let $a_1, b_1, c_1$ and $a_2, b_2, c_2$ be the direction ratios of $\vec{b_1}$ and $\vec{b_2}$, respectively. The given lines are coplanar if and only if,
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$
50. Two lines $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ are coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$
51. The equation of the plane containing the lines $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ is
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$
52. If $\vec{n_1}$ and $\vec{n_2}$ are normals to the planes $\vec{r}.\vec{n_1}$ = $d_1$ and $\vec{r}.\vec{n}_2$ = d2, and $\theta$ is the angle between the normals drawn from some common point, then $\cos\theta=\left|\dfrac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}\right|$.
53. Let $\theta$ be the angle between two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$.
The direction ratios of the normal to the planes are $A_1, B_1, C_1$ and $A_2, B_2, C_2$.
$$\cos\theta=\left|\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|$$

54. The angle $\theta$ between the line and the normal to the plane is given by

$$\cos\left(\frac{\pi}{2}-\theta\right)-\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|} \Rightarrow \sin\theta=\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

55. The distance of point P with position vector $\vec{a}$ from a plane $\vec{r}\cdot\vec{N}=d$ is $\frac{|\vec{a}\cdot\vec{N}-d|}{|\vec{N}|}$, where $\vec{N}$ is the normal to the plane.

56. The length of the perpendicular from the origin O to the plane $\vec{r}\cdot\vec{N}=d$ is $\frac{|d|}{|\vec{N}|}$, where $\vec{N}$ is the normal to the plane.

**Class : 12th Maths**
**Chapter- 11 : Three dimensional Geometry**

## Three dimensional Geometry

### Shortest distance between

(i) two skew lines is the line segment perpendicular to both the lines

(ii) $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is $\left|\dfrac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$

(iii) $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ is

$$\dfrac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

(iv) Parallel line $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is $\dfrac{|\vec{b}\times(\vec{a}_2-\vec{a}_1)|}{|\vec{b}|}$

### Direction ratios and direction cosines of line

D. Cs of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes. If $l, m, n$ are the D. Cs of a line, then $l^2+m^2+n^2=1$. D. Cs of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\dfrac{x_2-x_1}{PQ}, \dfrac{y_2-y_1}{PQ}, \dfrac{z_2-z_1}{PQ}$, where $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

D.Rs of a line are the no.s which are proportional to the D.Cs of the line if $l, m, n$ are the D.Cs and $a, b, c$ are D.Rs of a line, then

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\frac{b}{\sqrt{a^2+b^2+c^2}},\ n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

### Equation of a plane

(i) which is at distance 'd' from origin and D.C.s of the normal to the plane as $l,m,n$ is $lx+my+nz=d$.

(ii) $\perp r$ to a given line with D.Rs. $A,B,C$ and passing through $(x_1,y_1,z_1)$ is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

(iii) Passing through three non-collinear points $(x_1,y_1,z_1)$ $(x_2,y_2,z_2)$, $(x_3,y_3,z_3)$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$.

### Vector equation of a plane

(i) which contains three non-collinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$.

(ii) That passes through the intersection of planes $\vec{r}\cdot\vec{n}_1=d_1$ & $\vec{r}\cdot\vec{n}_2=d_2$ is $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, $\lambda$ – non-zero constant.

### Skew lines

These are the lines in space which are neither parallel nor intersecting. They lie in different planes. Angle between skew lines is the angle between two intersecting lines drawn from any point (origin) parallel to each of the skew lines.

### Angle between the two lines

if $l_1, m_1, n_1$, $l_2, m_2, n_2$ are the D.Cs and $a_1, b_1, c_1, a_2, b_2, c_2$ are the D.Rs of the two lines and 'θ' is the acute angle between them, then

$$\cos\theta=|l_1l_2+m_1m_2+n_1n_2|=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

### Characteristics of planes

Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are coplanar if $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$. Equation of a plane that cuts co-ordinate axes at (a,0,0), (0,b,0), (0,0,c) is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.

The distance of a point with postion vector $\vec{a}$ from the plane $\vec{r}\cdot\hat{n}=d$ is $|d-\vec{a}\cdot\hat{n}|$. The distance from a point $(x_1, y_1, z_1)$ to the plane $Ax+By+Cz+D=0$ is $\left|\dfrac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$

### Line of vector equations in 3D

**Vector equation of a line** passing through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$

**Vector equation of a line** which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

**Equation of a line** through point $(x_1, y_1, z_1)$ and having D.Cs $l, m, n$ is $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ Also, equation of a line that passes through two points.

### Angle between two lines

If 'θ' is the acute angle between $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ then, $\cos\theta=\left|\dfrac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1|\cdot|\vec{b}_2|}\right|$

if $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ and $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$ are the equations of two lines, then acute angle between them is $\cos\theta=|l_1\cdot l_2+m_1\cdot m_2+n_1\cdot n_2|$

# Important Questions

## Multiple Choice questions-

1. Distance between two planes:
   2x + 3y + 4z = 5 and 4x + 6y + 8z = 12 is
   (a) 2 units
   (b) 4 units
   (c) 8 units
   (d) $\frac{1}{\sqrt{29}}$ units.

2. The planes 2x – y + 4z = 3 and 5x – 2.5y +10 z = 6 are
   (a) perpendicular
   (b) parallel
   (c) intersect along y-axis
   (d) passes through $\left(0, 0, \frac{5}{4}\right)$

3. The co-ordinates of the foot of the perpendicular drawn from the point (2, 5, 7) on the x-axis are given by:
   (a) (2, 0, 0)
   (b) (0, 5, 0)
   (c) (0, 0, 7)
   (d) (0, 5, 7).

4. If α, ß, γ are the angles that a line makes with the positive direction of x, y, z axis, respectively, then the direction-cosines of the line are:
   (a) < sin α, sin ß, sin γ >
   (b) < cos α, cos ß, cos γ >
   (c) < tan α, tan ß, tan γ >
   (d) < $\cos^2$ α, $\cos^2$ ß, $\cos^2$ γ >.

5. The distance of a point P (a, b, c) from x-axis is
   (a) $\sqrt{a^2 + c^2}$
   (b) $\sqrt{a^2 + b^2}$
   (c) $\sqrt{b^2 + c^2}$
   (d) $b^2 + c^2$.

6. If the direction-cosines of a line are < k, k, k >, then
   (a) k > 0
   (b) 0 < k < 1
   (c) k = 1
   (d) k = $\frac{1}{\sqrt{3}}$ or - $\frac{1}{\sqrt{3}}$

7. The reflection of the point (α, ß, γ) in the xy-plane is:
   (a) (α, ß, 0)
   (b) (0, 0, γ)
   (c) (-α, -ß, γ)
   (d) (α, ß, -γ).

8. What is the distance (in units) between two planes:
   3x + 5y + 7z = 3 and 9x + 15y + 21z = 9?
   (a) 0
   (b) 3
   (c) $\frac{6}{\sqrt{83}}$
   (d) 6.

9. The equation of the line in vector form passing through the point (-1, 3, 5) and parallel to line $\frac{x-3}{2} = \frac{y-4}{3}$, z = 2 is
   (a) $\vec{r} = (-\hat{\imath} + 3\hat{\jmath} + 5\hat{k}) + \lambda(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$
   (b) $\vec{r} = (-\hat{\imath} + 3\hat{\jmath} + 5\hat{k}) + \lambda(2\hat{\imath} + 3\hat{\jmath})$
   (c) $\vec{r} = (2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}) + \lambda(-\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$
   (d) $\vec{r} = (2\hat{\imath} + 3\hat{\jmath}) + \lambda(-\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

10. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z-2}{2}$ lie in the plane x + 3y – αz + ß = 0. Then (α, ß) equals:
    (a) (-6, -17)
    (b) (5, -15)ss
    (c) (-5, 5)
    (d) (6, -17).

## Very Short Questions:

1. Find the acute angle which the line with direction-cosines $< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, n >$ makes with positive direction of z-axis.
2. Find the direction-cosines of the line.
   $$\frac{x-1}{2} = -y = \frac{z+1}{2}$$
3. If α, β, γ are direction-angles of a line, prove that cos 2a + cos 2P + cos 2y +1 = 0.
4. Find the length of the intercept, cut off by the plane 2x + y – z = 5 on the x-axis.
5. Find the length of the perpendicular drawn from the point P(3, -4,5) on the z-axis.
6. Find the vector equation of a plane, which is at a distance of 5 units from the origin and whose normal vector is $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$
7. If a line makes angles 90°, 135°, 45° with the x,y and z-axes respectively, find its direction cosines.
8. Find the co-ordinates of the point where the line through the points A (3,4,1) and B (5,1, 6) crosses the xy-plane.

9. Find the vector equation of the line which passes through the point (3,4,5) and is parallel to the vector $2\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

## Short Questions:

1. Find the acute angle between the lines whose direction-ratios are:
   < 1,1,2 > and <-3, -4,1 >.
2. Find the angle between the following pair of lines:
   $\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$
   and check whether the lines are parallel or perpendicular.
3. Find the vector equation of the line joining (1.2.3) and (-3,4,3) and show that it is perpendicular to the z-axis.
4. Find the vector equation of the plane, which is $\frac{6}{\sqrt{29}}$ at a distance of units from the origin and its normal vector from the origin is $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$. Also, find its cartesian form.
5. Find the direction-cosines of the unit vector perpendicular to the plane $\vec{r} \cdot (6\hat{\imath} - 3\hat{\jmath} - 2\hat{k}) + 1 = 0$ through the origin.
6. Find the acute angle between the lines
   $\frac{x-4}{3} = \frac{y+3}{4} = \frac{z+1}{5}$ and $\frac{x-1}{4} = \frac{y+1}{-3} = \frac{z+10}{5}$
7. Find the angle between the line:
   $\vec{r} = (\hat{\imath} - \hat{\jmath} + \hat{k}) + \lambda(2\hat{\imath} - \hat{\jmath} + 3\hat{k})$ and the plane $\vec{r} \cdot (2\hat{\imath} + \hat{\jmath} - \hat{k}) = 4$ Also, find whether the line is parallel to the plane or not .
8. Find the value of 'λ', so that the lines:
   $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not

## Long Questions:

1. Find the shortest distance between the lines: $\vec{r} = (4\hat{\imath} - \hat{\jmath}) + \lambda(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$ and $\vec{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \mu(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$
2. A line makes angles α, β, γ, δ with the four diagonals of a cube, prove that:
   $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.
3. Find the equation of the plane through the line $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$ and parallel to the line:
   $\frac{x+1}{2} = \frac{1-y}{4} = \frac{z+2}{1}$
   Hence, find the shortest distance between the lines.
4. Find the Vector and Cartesian equations of the plane passing through the points (2, 2, -1), (3,4,2) and (7,0,6). Also, find the vector equation of a plane passing through (4,3,1) and parallel to the plane obtained above.

## Case Study Questions:

**1.** Suppose the floor of a hotel is made up of mirror polished Kota stone. Also, there is a large crystal chandelier attached at the ceiling of the hotel. Consider the floor of the hotel as a plane having equation x - 2y + 2z = 3 and crystal chandelier at the point (3, -2, 1).

Based on the above information, answer the following questions.

(i) The d.r'.s of the perpendicular from the point (3, -2, 1) to the plane x - 2y + 2z = 3, is:
   a. < 1, 2, 2 >
   b. < 1, -2, 2 >
   c. < 2, 1, 2 >
   d. < 2, -1, 2 >

(ii) The length of the perpendicular from the point (3, -2, 1) to the plane x - 2y + 2z = 3, is:
   a. $\frac{2}{3}$ units
   b. 3 units
   c. 2 units
   d. None of these

(iii) The equation of the perpendicular from the point (3, -2, 1) to the plane x - 2y + 2z = 3, is:
   a. $\frac{x-3}{1} = \frac{y-2}{-2} = \frac{z-1}{2}$
   b. $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-1}{2}$
   c. $\frac{x+3}{1} = \frac{y+2}{-2} = \frac{z-1}{2}$
   d. None of these

(iv) The equation of plane parallel to the plane x - 2y + 2z = 3, which is at a unit distance from the point (3, -2, 1) is:

a. x - 2y + 2z = 0

b. x - 2y + 2z = 6

c. x - 2y + 2z = 12

d. Both (b) and (c)

(v) The image of the point (3, -2, 1) in the given plane is:

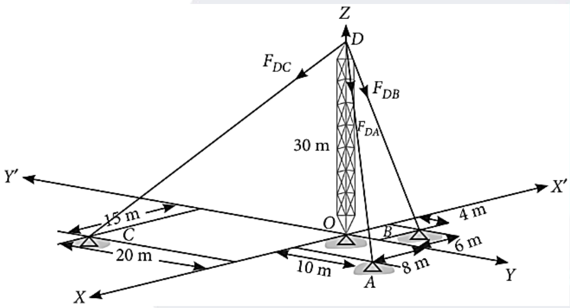
a. $\left(\frac{5}{3}, \frac{2}{3}, \frac{-5}{3}\right)$

b. $\left(\frac{-5}{3}, \frac{-2}{3}, \frac{5}{3}\right)$

c. $\left(\frac{-5}{3}, \frac{2}{3}, \frac{5}{3}\right)$

d. None of these

**2.** Consider the following diagram, where the forces in the cable are given.




Based on the above information, answer the following questions.

i. The equation of line along the cable AD is:

a. $\frac{x}{5} = \frac{y}{4} = \frac{z-30}{15}$

b. $\frac{x}{4} = \frac{y}{5} = \frac{z-30}{15}$

c. $\frac{x}{5} = \frac{y}{4} = \frac{30-z}{15}$

d. $\frac{x}{4} = \frac{y}{5} = \frac{30-z}{15}$

ii. The length of cable DC is:

a. $4\sqrt{61}\, m$

b. $5\sqrt{61}\, m$

c. $6\sqrt{61}\, m$

d. $7\sqrt{61}\, m$

iii. The vector DB is:

a. $-6\hat{i} + 4\hat{j} - 30\hat{k}$

b. $6\hat{i} - 4\hat{j} + 30\hat{k}$

c. $6\hat{i} + 4\hat{j} + 30\hat{k}$

d. None of these

iv. The sum of vectors along the cables is:

a. $17\hat{i} + 6\hat{j} + 90\hat{k}$

b. $17\hat{i} - 6\hat{j} - 90\hat{k}$

c. $17\hat{i} + 6\hat{j} - 90\hat{k}$

d. None of these

v. The sum of distances of points A, B and C from the origin, i.e., OA + OB + OC is:

a. $\sqrt{164} + \sqrt{52} + \sqrt{625}$

b. $\sqrt{52} + \sqrt{625} + \sqrt{48}$

c. $\sqrt{164} + \sqrt{625} + \sqrt{49}$

d. None of these

# Answer Key

## Multiple Choice Questions-

1. **Answer:** (d) $\frac{1}{\sqrt{29}}$ units.
2. **Answer:** (b) parallel
3. **Answer:** (a) (2, 0, 0)
4. **Answer:** (b) < cos α, cos ß, cos γ >
5. **Answer:** © $\sqrt{b^2 + c^2}$
6. **Answer:** © k = 1
7. **Answer:** (d) (α, ß, -γ).
8. **Answer:** (a) 0
9. **Answer:** (b) $\vec{r} = (-\hat{i} + 3\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 3\hat{j})$
10. **Answer:** (a) (-6, -17)

## Very Short Answer:

**1. Solution:**

$l^2 + m^2 + n^2 = 1$

$$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{16}}\right)^2 + n^2 = 1$$

$$\Rightarrow \frac{1}{3} + \frac{1}{6} + n^2 = 1$$

$n^2 = 1 - \frac{1}{2}$

$n^2 = \frac{1}{2}$

$n = \frac{1}{\sqrt{2}}$

Thus, $\cos \alpha = \frac{1}{\sqrt{2}}$

Hence, $\alpha = 45°$ or $\frac{\pi}{4}$

**2. Solution:**

The given line is $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+1}{2}$

Its direction-ratios are <2, -1, 2>.

Hence, its direction- cosine are:

$$< \frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{2}{\sqrt{4+1+4}} >$$

$$i.e., < \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} > \text{or} < \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} >$$

**3. Solution:**

Since α, β, γ are direction-angles of a line,

∴ cos2 α + cos2 β + cos2γ = 1

$$< \frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{2}{\sqrt{4+1+4}} >$$

$$i.e., < \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} > \text{or} < \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} >$$

⇒ 1 + cos2α + 1 + cos2β + 1 + cos2γ = 2

⇒ cos 2α + cos 2β + cos 2γ + 1 = 0, which is true.

**4. Solution:**

The given plane is2x + y – z = 5

$$\Rightarrow \frac{x}{5/2} + \frac{y}{5} + \frac{z}{-5} = 1$$

Its intercepts are $\frac{x}{5/2}$, 5 and -5.

Hence, the length of the intercept on the x-axis is $\frac{x}{5/2}$

**5. Solution:**

Length of the perpendicular from P (3, -4,5) on the z-axis

$$= \sqrt{(3)^2 + (-4)^2}$$

$$= \sqrt{9+16} = \sqrt{25} = 5 \text{ units}$$

**6. Solution:**

Let $\vec{n} = 2\hat{i} - \hat{j} + 2\hat{k}$

Then, $|\vec{n}| = \sqrt{4+1+4} = \sqrt{9} = 3$

Now, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - \hat{j} + 2\hat{k}}{3}$

Hence, the reqd. equation of the plane is:

$$\vec{r} \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 5$$

$$\Rightarrow \vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 15.$$

**7. Solution:**

Direction cosines of the line are:

< cos 90°, cos 135°, cos 45° >

$$< 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

**8. Solution:**

The equations of the line through A (3,4,1) and B (5,1,6) are:

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1}$$

$$\Rightarrow \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5} \quad ....(1)$$

Any point on (1) is (3 + 2k,4- 3k, 1 + 5k) .... (2)

This lies on xy-plane (z = 0).

∴ 1 + 5k = 0 ⇒ k = $-\frac{1}{5}$

Putting in (2), [ 3 - $\frac{2}{5}$, 4 + $\frac{3}{5}$, 1-1)

i.e. ($\frac{13}{5}$, $\frac{23}{5}$, 0)

which are the reqd. co-ordinates of the point.

**9. Solution:**

The vector equation of the line is $\vec{r} = \vec{a} + \lambda\vec{m}$

i.e., $\vec{r} = (3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + \lambda(2\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$

## Short Answer:

**1. Solution:**

$$\cos\theta \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{|(1)-(-3)+(1)(-4)+(2)(1)|}{\sqrt{1+1+4}\sqrt{9+16+1}}$$

$$\frac{|-3-4+2|}{\sqrt{6}\sqrt{26}} = \frac{5}{\sqrt{156}}$$

Hence, $\theta = \cos^{-1}\left(\frac{5}{\sqrt{156}}\right)$.

**2. Solution:**

The given lines can be rewritten as:

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \quad ....(1)$$

$$\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4} \quad ....(2)$$

Here < 2,7, – 3 > and < -1,2,4 > are direction- ratios of lines (1) and (2) respectively.

$\therefore\ \cos\theta=\dfrac{(2)(-1)+(7)(2)+(-3)(4)}{\sqrt{4+49+9}\ \sqrt{1+4+16}}$

$=\dfrac{-2+14-12}{\sqrt{62}\ \sqrt{21}}=0$

$\Rightarrow\ \theta=\dfrac{\pi}{2}.$

Hence, the given lines are perpendicular.

**3. Solution:**

Vector equation of the line passing through (1, 2, 3) and (–3, 4, 3) is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

where $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=-3\hat{i}+4\hat{j}+3\hat{k}$

$\Rightarrow\ \vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(-4\hat{i}+2\hat{j})$ ....(1)

Equation of $z$-axis is $\vec{r}=\mu\hat{k}$ ....(2)

Since $(-4\hat{i}+2\hat{j})\cdot\hat{k}=0=0$

∴ Line (1) is perpendicular to z-axis.

**4. Solution:**

Let $\vec{n}=2\hat{i}-3\hat{j}+4\hat{k}$

Then $|\vec{n}|=\sqrt{4+9+16}=\sqrt{29}$

Now $\hat{n}=\dfrac{\vec{n}}{|\vec{n}|}=\dfrac{2\hat{i}-3\hat{j}+4\hat{k}}{\sqrt{29}}$

Hence, the reqd. equation of the plane is:

$\vec{r}\cdot\left(\dfrac{2}{\sqrt{29}}\hat{i}-\dfrac{3}{\sqrt{29}}\hat{j}+\dfrac{4}{\sqrt{29}}\hat{k}\right)=\dfrac{6}{\sqrt{29}}$

In Cartesian Form:

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot\left(\dfrac{2}{\sqrt{29}}\hat{i}-\dfrac{3}{\sqrt{29}}\hat{j}+\dfrac{4}{\sqrt{29}}\hat{k}\right)=\dfrac{6}{\sqrt{29}}$

$\Rightarrow\ (x)\left(\dfrac{2}{\sqrt{29}}\right)+y\left(\dfrac{-3}{\sqrt{29}}\right)+z\left(\dfrac{4}{\sqrt{29}}\right)=\dfrac{6}{\sqrt{29}}$

$\Rightarrow\ 2x-3y+4z=6$

**5. Solution:**

The give plane is $\vec{r}\cdot(6\hat{i}-3\hat{j}-2\hat{k})+1=0$

$\vec{r}\cdot(6\hat{i}-3\hat{j}-2\hat{k})=1$ ....(1)

Now $\left|-6\hat{i}+3\hat{j}+2\hat{k}\right|=\sqrt{36+9+4}$

$=\sqrt{49}=7$

Dividing (1) by 7,

$\vec{r}\cdot\left(-\dfrac{6}{7}\hat{i}+\dfrac{3}{7}\hat{j}+\dfrac{2}{7}\hat{k}\right)=\dfrac{1}{7}$

Which is the equation of the plane in the form $\vec{r}\cdot\hat{n}=p$

Thus, $\hat{n}=-\dfrac{6}{7}\hat{i}+\dfrac{3}{7}\hat{j}+\dfrac{2}{7}\hat{k}$

which is the unit vector perpendicular to the plane through the origin.

Hence, the direction – cosines of $\hat{n}$ are $<-\dfrac{6}{7},\dfrac{3}{7},\dfrac{2}{7}>$

**6. Solution:**

Vector in the direction of first line

$\dfrac{x-4}{3}=\dfrac{y+3}{4}=\dfrac{z+1}{5}$

$\vec{b}=(3\hat{i}+4\hat{j}+5\hat{k})$

Vector in the direction of second line

$\dfrac{x-1}{4}=\dfrac{y+1}{-3}=\dfrac{z+10}{5}$

$\vec{d}=4\hat{i}-3\hat{j}+5\hat{k}$

∴ θ, the angle between two given lines is given by:

$\cos\theta=\dfrac{\vec{b}\cdot\vec{d}}{|\vec{b}||\vec{d}|}$

$=\dfrac{(3\hat{i}+4\hat{j}+5\hat{k})\cdot(4\hat{i}-3\hat{j}+5\hat{k})}{|3\hat{i}+4\hat{j}+5\hat{k}||4\hat{i}-3\hat{j}+5\hat{k}|}$

$=\dfrac{(3)(4)+(4)(-3)+(5)(5)}{\sqrt{9+16+25}\ \sqrt{16+9+25}}$

$=\dfrac{12-12+25}{\sqrt{50}\ \sqrt{50}}=\dfrac{25}{50}=\dfrac{1}{2}$

Hence, $\theta=\dfrac{\pi}{3}$

**7. Solution:**

The given line is:

$\vec{r}=(\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}-\hat{j}+3\hat{k})$ and the given plane is $\vec{r}\cdot(2\hat{i}+\hat{j}-k)=4.$

Now the line is parallel to $2\hat{i}-\hat{j}+3\hat{k}$ and normal to the plane $2\hat{i}+\hat{i}-\hat{k}$

If 'θ' is the angle between the line and the plane, then $\left(\dfrac{\pi}{2}-\theta\right)$ is the angle between the line and normal to the plane.

Then $\cos\left(\dfrac{\pi}{2}-\theta\right)=\dfrac{(2\hat{i}-\hat{j}+3\hat{k})\cdot(2\hat{i}+\hat{j}-\hat{k})}{\sqrt{4+1+9}\ \sqrt{4+1+1}}$

$\Rightarrow\ \sin\theta=\dfrac{4-1-3}{\sqrt{14}\ \sqrt{6}}=0$

$\Rightarrow\ \theta=0°.$

Hence, the line is parallel to the plane.

**8. Solution:**

(i) The given lines are

$$\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2} \quad ....(1)$$

and $$\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5} \quad ....(2)$$

These are perpendicular if:

$$(-3)\left(-\frac{3\lambda}{7}\right)+\left(\frac{\lambda}{7}\right)(1)+2(-5)=0$$

if $\frac{9\lambda}{7}+\frac{\lambda}{7}-10=0$ if $\frac{10\lambda}{7}=10.$

Hence λ = 1.

(ii) The direction cosines of line (1) are <-3,1,2>

The direction cosines of line (2) are < -3,1, -5 >

Clearly, the lines are intersecting.

## Long Answer:

**1. Solution:**

Comparing given equations with:

$\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ *and* $\vec{r}=\vec{a_2}+\mu\vec{b_2}$,

We have: $\vec{b_1}=\vec{i}+\vec{j}-3\hat{k}, \vec{b_2}=2\hat{i}+4\vec{j}-5\hat{k}$

and $\vec{a_1}=(4\hat{i}-\hat{j}), \vec{a_2}=\hat{i}-\hat{j}+2\hat{k}$

Now, $\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}$

$=\hat{i}(-10+12)-\hat{j}(-5+6)+\hat{k}(4-4)=2\hat{i}-\hat{j}$

$\therefore |\vec{b_1}\times\vec{b_2}|=\sqrt{(2)^2+(-1)^2+0^2}=\sqrt{4+1+0}=\sqrt{5}$

Also, $\vec{a_2}-\vec{a_1}=(\hat{i}-\hat{j}+2\hat{k})-(4\hat{i}-\hat{j})=-3\hat{i}+2\hat{k}$

$\therefore d$, the S.D. $=\left|\frac{(\vec{b_1}\times\vec{b_2})\cdot(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right|$

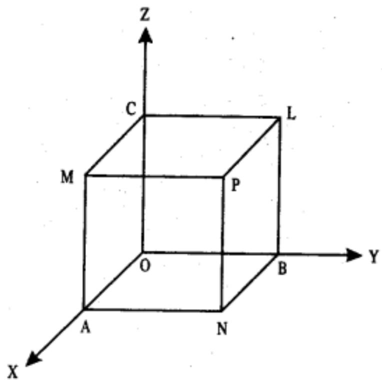
$=\left|\frac{(2\hat{i}-\hat{j})\cdot(-3\hat{i}+2\hat{k})}{\sqrt{5}}\right|$

$=\left|\frac{(2)(-3)+(-1)(0)+(0)(2)}{\sqrt{5}}\right|$

$=\left|\frac{-6-0+0}{\sqrt{5}}\right|=\frac{6}{\sqrt{5}}=\frac{6\sqrt{5}}{5}$ units

**2. Solution:**

Let O be the origin and OA, OB, OC (each = a) be the axes.




Thus the co-ordinates of the points are :

O (0,0,0), A (a, 0,0), B (0, a, 0), C (0,0, a),

P (a, a, a), L (0, a, a), M (a, 0, a), N (a, a, 0).

Here OP, AL, BM and CN are four diagonals.

Let < l, m, n > be the direction-cosines of the given line.

Now direction-ratios of OP are:

<a-0,a-0,a-0>i.e.<a,a,a>

i.e. < 1,1,1 >,

direction-ratios of AL are:

<0-a, a-0, a-0> i.e. <-a,a,a>

i.e. <-l, 1,1 >,

direction-ratios of BM are:

<a-0,0-a, a-0>

i.e. <a,-a,a> i.e. < 1,-1, 1 >

and direction-ratios of CN are:

<a-0,a-0,0-a> i.e. <a,a,-a>

i.e. < 1,1,-1 >.

Thus the direction-cosines of OP are:

$$<\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}>$$

the direction-cosines of AL are:

$$<-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}>$$

the direction-cosines of BM are:

$$<\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}>$$

and the direction-cosines of CN are:

$$<\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}>$$

If the given line makes an angle 'a' with OP, then:

$$\cos\alpha=\left|l\left(\frac{1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)\right|$$

$$\therefore \cos\alpha=\frac{|l+m+n|}{\sqrt{3}} \quad ....(1)$$

If the given line makes an angle 'β' with AL, then:

$$\cos\beta = \left| l\left(-\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|$$

$$\therefore \cos\beta = \frac{|-l+m+n|}{\sqrt{3}} \quad ....(2)$$

$$\text{Similarly, } \cos\gamma = \frac{|l+m+n|}{\sqrt{3}} \quad ....(3)$$

$$\text{and } \cos\delta = \frac{|l+m+n|}{\sqrt{3}} \quad ....(4)$$

Squaring and adding (1), (2), (3) and (4), we get:

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$\frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)] = \frac{1}{3}[4(1)]$$

$$\text{Hence, } \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

**3. Solution:**

The two given lines are:

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2} \quad ....(1)$$

$$\text{and } \frac{x+1}{2} = \frac{1-y}{4} = \frac{z+2}{1} \quad ....(2)$$

Let <a, b, c> be the direction-ratios of the normal to the plane containing line (1).

∴ Equation of the plane is:

a(x- l) + b(y-4) + c(z-4) ...(3),

where 3a + 2b – 2c = 0 ...(4)

[∵ Reqd. plane contains line (1)] and 2a – 4b + 1.c = 0

[∵ line (1) a parallel to the reqd. plane] Solving (4) and (5),

$$\Rightarrow \frac{a}{-6} = \frac{b}{-7} = \frac{c}{-16}$$

$$\Rightarrow \frac{a}{6} = \frac{b}{7} = \frac{c}{16} = k, \text{ where } k \neq 0.$$

$$\therefore a = 6k, b = 7k \text{ and } c = 16k$$

Putting in (3),

6k(x- 1) + 7k(y – 4) + 16k(z – 4) = 0

= 6(x – 1) + 7(y – 4) + 16(z – 4) =0

[∵k ≠ 0]

⇒ 6x + 7y+ 16z-98 = 0,

which is the required equation of the plane.

Now, S.D. between two lines = perpendicular distance of (-1,1, – 2) from the plane

$$i.e., S.D. = \left| \frac{6(-1)+7(1)+16(-2)-98}{\sqrt{(6)^2+(7)^2+(16)^2}} \right|$$

6(-1) + 7(1) +16(-2) – 98

V(6)² + (7)² + (16)²

-6 + 7-32-98 V³⁶ + 49 + 256

**4. Solution:**

(i) Cartesian equations

Any plane through (2,2, -1) is:

a(x – 2) + b(y- 2) + c(z + 1) = 0 ....(1)

Since the plane passes through the points (3,4,2) and (7,0,6),

∴ a(3 – 2) + b(4 – 2) + c(2 +1) = 0

and a(7 – 2) + b(0 – 2) + c(6 + 1) = 0

⇒ a + 2b + 3c = 0 ....(2)

and 5a – 2b + 7c = 0 ....(3)

$$\text{Solving (2) and (3), } \frac{a}{14+6} = \frac{b}{15-7} = \frac{c}{-2-10}$$

$$\Rightarrow \frac{a}{20} = \frac{b}{8} = \frac{c}{-12}$$

$$\Rightarrow \frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = k \text{ (say), value } k \neq 0$$

∴ a = 5k,b = 2k and c = -3k,

Putting the values of a, b, c in (1), we get:

5k(x – 2) + 2k(y – 2) – 3k(z + 1) = 0

⇒ 5(x-2) + 2(y-2)-3(z+ 1) =0[∵ k ≠ 0]

=» 5x- 10 + 2y-4-3z-3 = 0

=» 5x + 2y-3z-17 = 0, ....(4)

which is the reqd. Cartesian equation.

Its vector equation is $\vec{r}\cdot(5\hat{i}+2\hat{j}-3\hat{k}) = 17$.

(ii) Any plane parallel to (4) is

5x + 2y – 3z + λ – 0 ... (5)

Since it passes through (4, 3,1),

5(4) + 2(3) – 3(1) + λ = 0

⇒ 20 + 6 — 3 + λ = 0

⇒ λ = -23.

Putting in (5), 5x + 2y – 3z – 23 = 0, which is the reqd. equation.

Its vector equation is $\vec{r}\cdot(5\hat{i}+2\hat{j}-3\hat{k}) = 23$.

## Case Study Answers:

**1. Answer:**

i. (b) <1, –2, 2>

**Solution:**

Equation of plane is *x* – 2*y* = 3

∴ D.R.'s of normal to the plane are, which is also the D.R.'s of perpendicular from the point (3, –2, 1) to the given plane.

(ii) (c) 2 units

**Solution:**

Required length = Perpendicular distance from (3, –2, 1) to the plane $x - 2y + 2z = 3$

$$= \left|\frac{3-2(-2)+2(1)-3}{\sqrt{1^2+(-2)^2+2^2}}\right| = \frac{6}{3} = 2 \text{ units}$$

(iii) (b) $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-1}{2}$

**Solution:**

The equation of perpendicular from the point $(x_1, y_1, z_1)$ to the plane $ax + by + cz = d$ is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Here, $x_1 = 3, y_1 = -2, z_1 = 1$ and $a = 1, b = -2, c = 2$

$\therefore$ Required equation is $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-1}{2}$

(iv) (d) Both (b) and (c)

**Solution:**

The equation of the plane parallel to the plane $x - 2y + 2i - 3 = 0$ is $x - 2y + 2z + l = 0$

Now, distance of this from the point (3, –2, 1) is

$$= \left|\frac{3+4+2+\lambda}{\sqrt{1^2+(-2)^2+2^2}}\right| = \left|\frac{9+\lambda}{3}\right|$$

But, this distance is given to be unity

$\therefore |9+\lambda| = 3$

$\Rightarrow \lambda + 9 = \pm 3 \Rightarrow \lambda = -6$ or $-12$

Thus, required equation of planes are

$x - 2y + 2z - 6 = 0$ or $x - 2y + 2z - 12 = 0$

(v) (a) $\left(\frac{5}{3}, \frac{2}{3}, \frac{-5}{3}\right)$

**Solution:**

Let the coordinate of image of (3, –2, 1) be $Q(r + 3, -2r - 2, 2r + 1)$

Let $R$ be the mid-point of $PQ$, then coordinate of $R$ be

$$\left(\frac{r+6}{2}, \frac{-2r-4}{2}, r+1\right)$$

Since, $R$ lies on the plane $x - 2y + 2z = 3$

$$\therefore \left(\frac{r+6}{2}\right) - 2\left(\frac{-2r-4}{2}\right) + 2(r+1) = 3$$

$$\Rightarrow 9r = -12 \Rightarrow r = -\frac{4}{3}$$

Thus, the coordinates of Q be $\left(\frac{5}{3}, \frac{2}{3}, \frac{-5}{3}\right)$.

**2. Answer:**

i. (d) $\frac{x}{4} = \frac{y}{5} = \frac{30-z}{15}$

**Solution:**

Clearly, the coordinates of A are (8, 10, 0) and D are (0, 0, 30)

$$\frac{x-0}{8-0} = \frac{y-0}{10-0} = \frac{30-z}{-30}$$

$$\Rightarrow \frac{x}{4} = \frac{y}{5} = \frac{30-z}{15}$$

ii. (b) $5\sqrt{61}\ m$

**Solution:**

The coordinates of point care (15, –20, 0) and (0, 0, 30)

$\therefore$ Length of the cable DC

$$= \sqrt{(0-15)^2 + (0-20)^2 + (30-0)^2}$$

$$= \sqrt{225+400+900}$$

$$= \sqrt{1525} = \sqrt{61}\ m$$

iii. (a) $-6\hat{i} + 4\hat{j} - 30\hat{k}$

**Solution:**

Since, the coordinates of point B are (–6, 4, 0) and D are (0, 0, 30), therefore vector DB is

$(-6-0)\hat{i} + (4-0)\hat{j} + (0-30)\hat{k}$,

*i.e.*, $-6\hat{i} + 4\hat{j} - 30\hat{k}$

iv. (b) $17\hat{i} - 6\hat{j} - 90\hat{k}$

**Solution:**

Required sum

$(8\hat{i} + 10\hat{j} - 30\hat{k}) + (-6\hat{i} + 4\hat{j} - 30\hat{k})$

$+ (15\hat{i} - 20\hat{j} - 30\hat{k})\ 17\hat{i} - 6\hat{j} - 90\hat{k}$

v. (a) $\sqrt{164} + \sqrt{52} + \sqrt{625}$

**Solution:**

Clearly, $OA = \sqrt{8^2 + 10^2} = \sqrt{164}$

$OB = \sqrt{6^2 + 4^2} = \sqrt{36+16} = \sqrt{52}$

and $OC = \sqrt{15^2 + 20^2} = \sqrt{225+400} = \sqrt{625}$

# Linear Programming 12

## Top Concepts

1. Linear programming is the process of taking various linear inequalities relating to some situation and finding the best value obtainable under those conditions.
2. Linear programming is part of a very important branch of mathematics called 'Optimization Techniques'.
3. Problems which seek to maximise (or minimise) profit (or cost) form a general class of problems called optimisation problems.
4. A problem which seeks to maximise or minimise a linear function, subject to certain constraints as determined by a set of linear inequalities, is called an optimisation problem.
5. A linear programming problem may be defined as the problem of maximising or minimising a linear function subject to linear constraints. The constraints may be equalities or inequalities.
6. **Objective function:** The linear function $Z = ax + by$, where a and b are constants and x and y are decision variables, which has to be maximised or minimised is called a linear objective function. An objective function represents cost, profit or some other quantity to be maximised or minimised subject to constraints.
7. Linear inequalities or equations which are derived from the application problem are problem constraints.
8. Linear inequalities or equations or restrictions on the variables of a linear programming problem are called constraints.
9. The conditions $x \geq 0$ and $y \geq 0$ are called non-negative restrictions. Non-negative constraints are included because x and y are usually the number of items produced and one cannot produce a negative number of items. The smallest number of items one could produce is zero. These conditions are not (usually) stated; they are implied.
10. A linear programming problem is one which is concerned with finding the optimal value (maximum or minimum value) of a linear function (called objective function) of several variables (say x and y) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).
11. In **linear programming**, the term linear implies that all the mathematical relations used in the problem are linear while programming refers to the method of determining a particular program me or plan of action.
12. Forming a set of linear inequalities (constraints) for a given situation is called formulation of the linear programming problem (LPP).
13. **Mathematical formulation of linear programming problems**

    **Step I:** In every LPP, certain decisions are to be made. These decisions are represented by decision variables. These decision variables are those quantities whose values are to be determined. Identify the variables and denote them as $x_1, x_2, x_3$ .... or x, y and z etc.

    **Step II:** Identify the objective function and express it as a linear function of the variables introduced in Step I.

    **Step III:** In a LPP, the objective function may be in the form of maximising profits or minimising costs. Hence, identify the type of optimisation, i.e., maximisation or minimisation.

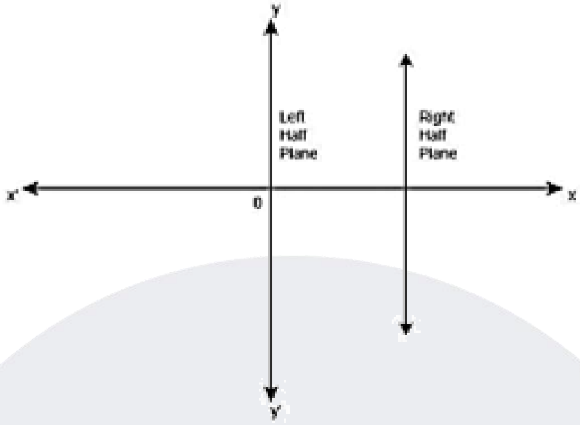
    **Step IV:** Identify the set of constraints stated in terms of decision variables and express them as linear inequations or equations as the case may be.

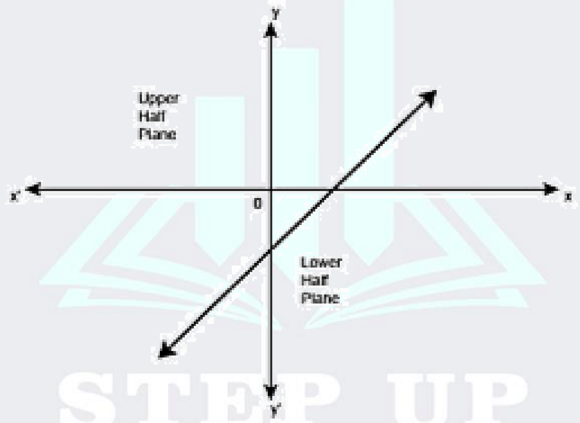
**Step V:** Add the non-negativity restrictions on the decision variables, as in the physical problems. Negative values of decision variables have no valid interpretation.

14. **General LPP is of the form**

Max (or min) $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, where $c_1, c_2, \ldots c_n$ are constants and $x_1, x_2, \ldots x_n$ are called decision variables such that $Ax \leq (\geq)B$ and $x_i \geq 0$.

15. A linear inequality in two variables represents a half plane geometrically. There are two types of half planes:

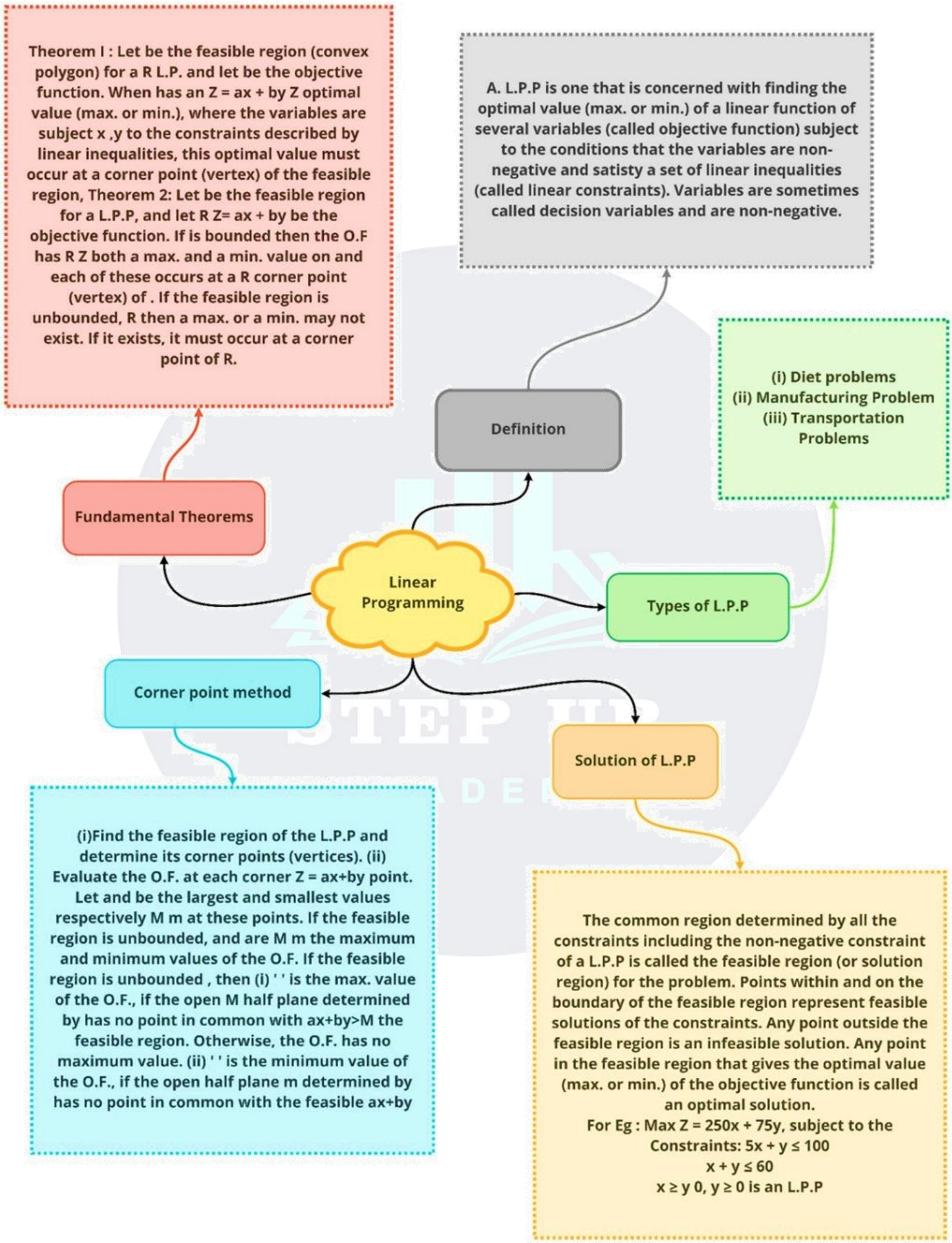



16. A set of values of the decision variables which satisfy the constraints of a linear programming problem is called a solution of LPP.
17. The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem. The region other than the feasible region is called the infeasible region.
18. Points within and on the boundary of the feasible region represent the feasible solution of the constraints.
19. Any point in the feasible region which gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.
20. Any point outside the feasible region is called an infeasible solution.
21. A corner point of a feasible region is the intersection of two boundary lines.
22. A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle.
23. **Corner Point Theorem 1:** Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point (vertex) of the feasible region.
24. **Corner Point Theorem 2:** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.
25. If R is unbounded, then a maximum or a minimum value of the objective function may not exist.

26. The graphical method for solving linear programming problems in two unknowns is as follows:
    **A.** Graph the feasible region.
    **B.** Compute the coordinates of the corner points.
    **C.** Substitute the coordinates of the corner points into the objective function to see which gives the optimal value.
    **D.** When the feasible region is bounded, M and m are the maximum and minimum values of Z.
    **E.** If the feasible region is not bounded, then this method can be misleading. Optimal solutions always exist when the feasible region is bounded but may or may not exist when the feasible region is unbounded.
    **i.** M is the maximum value of Z if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.
    **ii.** Similarly, m is the minimum value of Z if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.
27. Points within and on the boundary of the feasible region represent the feasible solutions of the constraints.
28. If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum for the function, then any point on the line segment joining these two points is also an optimal solution of the same type.
29. **Types of Linear Programming Problems**
    **i.** **Manufacturing problems:** Problems dealing with the determination of the number of units of different products to be produced and sold by a firm, when each product requires fixed manpower, machine hours and labour hour per unit of product, in order to make maximum profit.
    **ii.** **Diet problems:** Problems dealing with the determination of the amount of different kinds of nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrient.
    **iii.** **Transportation problems:** Problems dealing with the determination of the transportation schedule for the cheapest way to transport a product from plants/factories situated at different locations to different markets.
30. **Advantages of LPP**
    **i.** The linear programming technique helps to make the best possible use of the available productive resources (such as time, labour, machines etc.).
    **ii.** A significant advantage of linear programming is highlighting of such bottle necks.
31. **Disadvantages of LPP**
    **i.** Linear programming is applicable only to problems where the constraint and objective functions are linear, i.e., where they can be expressed as equations which represent straight lines.
    **ii.** Factors such as uncertainty, weather conditions etc. are not taken into consideration.

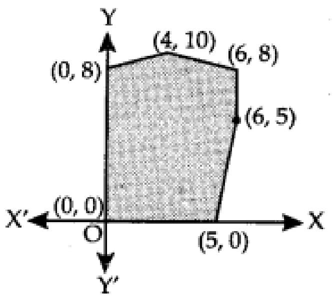
Class : 12th Maths
Chapter- 12 : Linear Programming



# Important Questions

## Multiple Choice Questions-

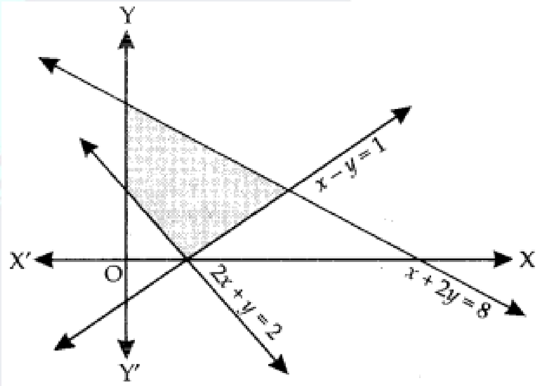
1. The point which does not lie in the half plane $2x + 3y - 12 < 0$ is
   (a) (1, 2)
   (b) (2, 1)
   (c) (2, 3)
   (d) (-3, 2).
2. The corner points of the feasible region determined by the following system of linear inequalities:
   $2x + y \le 10, x + 3y \le 15, x, y \ge 0$ are (0, 0), (5, 0), (3, 4) and (0, 5).
   Let $Z = px + qy$, where $p, q > 0$. Conditions on p and q so that the maximum of Z occurs at both (3, 4) and (0, 5) is
   (a) $p = 3q$
   (b) $p = 2q$
   (c) $p = q$
   (d) $q = 3p$.
3. The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5), (15, 15), (0, 20). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both the points (15, 15) and (0, 20) is
   (a) $p = q$
   (b) $p = 2q$
   (c) $q = 2p$
   (d) $q = 3p$.
4. The feasible solution for a LPP is shown in the following figure. Let $Z = 3x - 4y$ be the objective function.




   Minimum of Z occurs at:
   (a) (0, 0)
   (b) (0, 8)
   (c) (5, 0)
   (d) (4, 10).
5. The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5), (15, 15), (0, 20). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both the points (15, 15) and (0, 20) is Maximum of Z occurs at:
   (a) (5, 0)
   (b) (6, 5)
   (c) (6, 8)
   (d) (4, 10).

## Very Short Questions:

1. Draw the graph of the following LPP:
   $5x + 2y \le 10, x \ge 0, y \ge 0$.
2. Solve the system of linear inequations: $x + 2y \le 10$; $2x + y \le 8$.
3. Find the linear constraints for which the shaded area in the figure below is the solution set:




4. A small firm manufactures neclaces and bracelets. The total number of neclaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a neclace. The maximum number of hours available per day is 16. If the profit on a neclace is ₹100 and that on a bracelet is ₹300. Formulate an LPP for finding how many of each should be produced daily to maximize the profit?
   It is being given that at least one of each must be produced.
5. Old hens can be bought for? 2.00 each and young ones at? 5.00 each. The old hens lay 3 eggs per week and the young hens lay 5 eggs per week, each egg being worth 30 paise. A hen costs ₹1.00 per week to feed. A man has only ₹80 to spend for hens. Formulate the problem for maximum profit per week, assuming that he cannot house more than 20 hens.

## Long Questions:

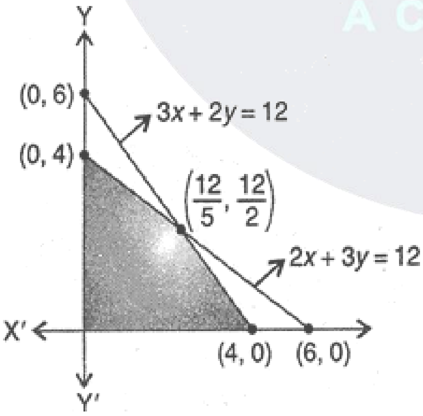
1. Maximize Z-5x + 3y subject to the constraints:
   $3x + 5y \le 15$, $5x + 2y \le 10$, $x \ge 0$, $y \ge 0$.
2. Minimize Z = 3x + 2y subject to the constraints:
   $x + y \ge 8$, $3x + 5y \le 15$, $x \ge 0$, $y \ge 0$.
3. Determine graphically the minimum value of the objective function:
   $Z = -50x + 20y$
   subject to the constraints:
   $2x - y \ge -5$, $3x + y \ge 3$, $2x - 3y \le 12$, $x, y \ge 0$.
   Hence, find the shortest distance between the lines.
2. Minimize and Maximize Z = 5x + 2y subject to the following constraints: $x - 2y \le 2$, $3x + 2y < 12$, $-3x + 2y \le 3$, $x \ge 0$, $y \ge 0$.

## Assertion and Reason Questions:

**1.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

Consider the graph of $2x+3y\le12$, $3x+2y\le12$, $x\ge0$, $y\ge0$.




**Assertion(A):** (5, 1) is an infeasible solution of the problem.

**Reason (R):** Any point inside the feasible region is called an infeasible solution.

**2.** Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

Consider the graph of constraints

$5x + y \le 100$, $x + y \le 60$, $x, y \ge 0$

**Assertion (A):** The points (10, 50), (0, 60) and (20, 0) are feasible solutions.

**Reason(R):** Points within and on the boundary of the feasible region represents infeasible solutions.

## Case Study Questions:

**1.** Suppose a dealer in rural area wishes to purpose a number of sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items for storage. An electronic sewing machine costs him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18.

Based on the above information, answer the following questions.

(i) Let x and y denotes the number of electronic sewing machines and manually operated sewing machines purchased by the dealer. If it is assume that the dealer purchased atleast one of the the given machines, then:

a. $X + y \ge 0$
b. $X + y < 0$
c. $X + y > 0$
d. $X + y \le 0$

(ii) Let the constraints in the given problem is represented by the following inequalities.

$x + y \le 20$

$360x + 240y \le 5760$

$x, y \ge 0$

Then which of the following point lie in its feasible region.

a. (0, 24)

b. (8, 12)

c. (20, 2)

d. None of these

(iii) If the objective function of the given problem is maximise z = 22x + 18y, then its optimal value occur at:

a. (0, 0)

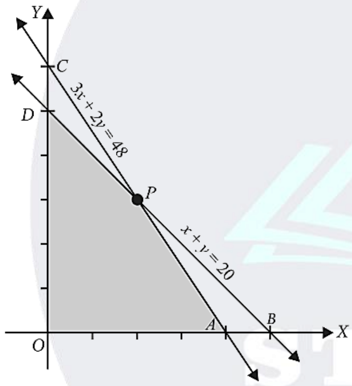
b. (16, 0)

c. (8, 12)

d. (0, 20)

(iv) Suppose the following shaded region APDO, represent the feasible region corresponding to mathematical formulation of given problem.

Then which of the following represent the coordinates of one of its corner points




a. (0, 24)

b. (12, 8)

c. (8, 12)

d. (6, 14)

(v) If an LPP admits optimal solution at two consecutive vertices of a feasible region, then:

a. The required optimal solution is at the midpoint of the tine joining two points.

b. The optimal solution occurs at every point on the tine joining these two points.

c. The LPP under consideration is not solvable.

d. The LPP under consideration must be reconstructed.

**2.** Corner points of the feasible region for an LPP are (0, 3), (5, 0), (6, 8), (0, 8). Let Z = 4x - 6y be the objective function.

Based on the above information, answer the following questions.

(i) The minimum value of Z occurs at:

a. (6, 8)

b. (5, 0)

c. (0, 3)

d. (0, 8)

(ii) Maximum value of Z occurs at:

a. (5, 0)

b. (0, 8)

c. (0, 3)

d. (6, 8)

(iii) Maximum of Z - Minimum of Z =

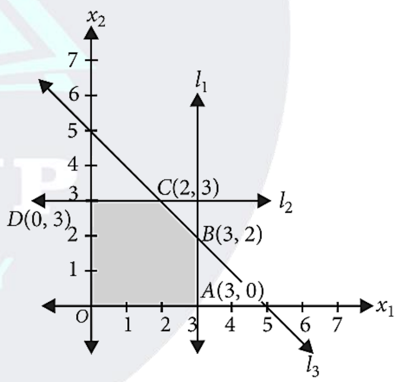
a. 58

b. 68

c. 78

d. 88

(iv) The corner points of the feasible region determined by the system of linear inequalities are:




a. (0, 0), (-3, 0), (3, 2). (2, 3)

b. (3, 0), (3, 2), (2, 3), (0, -3)

c. (0, 0), (3, 0), (3, 2), (2, 3), (0, 3)

d. None of these

(v) The feasible solution of LPP belongs to:

a. First and second quadrant.

b. First and third quadrant.

c. Only second quadrant.

d. Only first quadrant.

# Answer Key

## Multiple Choice questions-

1. **Answer:** (c) (2, 3)
2. **Answer:** (d) q = 3p.
3. **Answer:** (d) q = 3p.
4. **Answer:** (b) (0, 8)
5. **Answer:** (a) (5, 0)

## Very Short Answer:

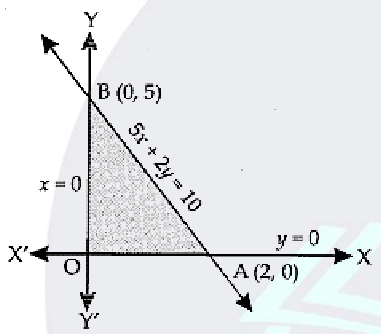
**1. Solution:**

Draw the line AB: 5.v + 2y = 10 ...(1),

which meets x-axis at A (2, 0) and y-axis at B (0,5).

Also, x = 0 is y-axis and y = 0 is x-axis.

Hence, the graph of the given LPP is as shown (shaded):

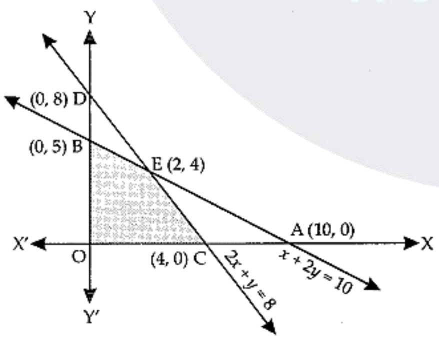



**2. Solution:**

Draw the st. lines x + 2y = 10 and 2x + y = 8.

These lines meet at E (2,4).

Hence, the solution of the given linear inequations is shown as shaded in the following figure:




**3. Solution:**

From the above shaded portion, the linear constraints are :

2x + y ≥ 2,x – y ≤ 1,

x + 2y ≤ 8, x ≥ 0, y ≥ 0.

**4. Solution:**

Let 'x' neclaces and 'y' bracelets be manufactured per day.

Then LPP problem is:

Maximize Z = 100x+300y

Subject to the constraints : x + y ≤ 24,

(1) (x) + $\frac{1}{2}$y ≤ l6,

i.e. 2x + y ≤ 32

and x ≥ 1

and y ≥ 1

i.e. x – 1 ≥ 0

and y – 1 ≥ 0.

**5. Solution:**

Let 'x' be the number of old hens and 'y' the number of young hens.

Profit = (3x + 5y) $\frac{30}{100}$ – (x + y) (1)

$= \frac{9x}{10} + \frac{3}{2}$ yx – y

$= \frac{y}{2} - \frac{x}{10} = \frac{5y-x}{10}$

∴ LPP problem is:

Maximize Z = $\frac{5y-x}{10}$ subject to:

x ≥ 0,

y ≥ 0,

x + y ≤ 20 and

2x + 5y ≤ 80.

## Long Answer:

**1. Solution:**

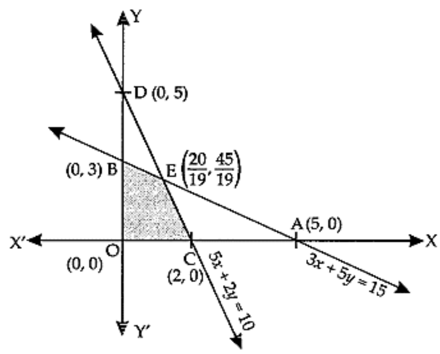
The system of constraints is :

3x + 5y ≤ 15 ...(1)

5x + 2y ≤ 10 ...(2)

and x ≥ 0, y≥ 0 ...(3)

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (3):




It is observed that the feasible region OCEB is bounded. Thus we use Corner Point Method to determine the maximum value of Z, where:

Z = 5x + 3y ...(4)

The co-ordinates of O, C, E and B are (0, 0), (2,0), $\left(\frac{20}{19}, \frac{45}{19}\right)$ (Solving 3x + 5y = 15 and 5x + 2y – 10) and (0, 3) respectively.

We evaluate Z at each comer point:

| Corner Point | Corresponding Value of Z |
|---|---|
| 0: (0, 0) | 0 |
| C: (2, 0) | 10 |
| E $\left(\left(\frac{20}{19}, \frac{45}{19}\right)\right)$ | $\frac{20}{19}$ (Maximum) |
| B (0, 3) | 9 |

Hence $Z_{max}$ = at the Point $\left(\frac{20}{19}, \frac{45}{19}\right)$

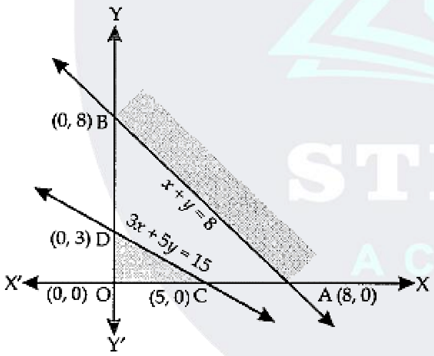
**2. Solution:**

The system of constraints is :

x +y ≥ 8, , x ≥ 0, y ≥ 0...(1)

3x + 5y ≤ 15 ...(2)

and x ≥ 0, y ≥ 0 ...(3)




It is observed that there is no point, which satisfies all (1) – (3) simultaneously.

Thus there is no feasible region.

Hence, there is no feasible solution.

Solution:

The system of constraints is :

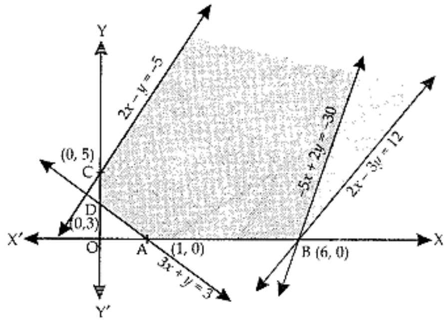
2x-y ≥ – 5 ...(1)

3x +y ≥ 3 ...(2)

2x – 3y ≤ 12 ...(3)

and x,y ≥ 0 ...(4)

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (4).




t is observed that the feasible region is unbounded.

We evaluate Z = – 50x + 20y at the corner points: A (1, 0), B (6, 0), C (0, 5) and D (0, 3):

| Corner Point | Corresponding Value of Z |
|---|---|
| A: (1, 0) | –50 |
| B: (6, 0) | –300 (Minimum) |
| C: (0, 5) | 100 |
| D: (0, 3) | 60 |

From the table, we observe that – 300 is the minimum value of Z.

But the feasible region is unbounded.

∴ – 300 may or may not be the minimum value of Z. "

For this, we draw the graph of the inequality.

– 50x + 20y < – 300

i.e. – 5x + 2y < – 30.

Since the remaining half-plane has common points with the feasible region,

∴ Z = – 50x + 20y has no minimum value.

**3. Solution:**

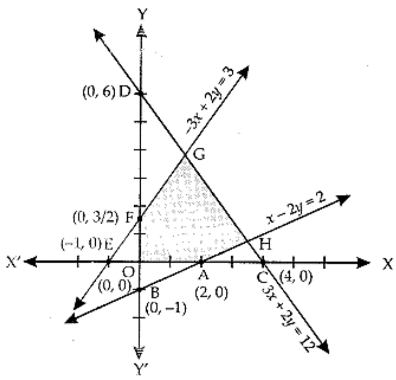
The given system of constraints is:

x – 2y ≤ 2 ...(1)

3x + 2y < 12 ...(2)

-3x + 2y ≤ 3 ...(3)

and x ≥ 0, y ≥ 0.



The shaded region in the above figure is the feasible region determined by the system of constraints (1) – (4). It is observed that the feasible region OAHGF is bounded. Thus we use Corner Point Method to determine the maximum and minimum value of Z, where

Z = 5x + 2y ...(5)

The co-ordinates of O, A, H, G and F are :

(0, 0). (2. 0), $\left(\frac{7}{2},\frac{3}{4}\right)$ and $\left(\frac{3}{2},\frac{15}{4},\frac{3}{2}\right)$

respectively. [Solving x

2y = 2 and 3x + 2y = 12 for

H and -3x + 2y = 3 and

3x + 2y = 12 for G]

We evaluate Z at each corner point:

| Corner Point | Corresponding Value of Z |
|---|---|
| 0: (0, 0) | 0 (Minimum) |
| A: (2, 0) | 10 |
| H $\left(\frac{7}{2},\frac{3}{4}\right)$ | 19 (Maximum) |
| G $\left(\frac{3}{2},\frac{15}{4}\right)$ | 15 |
| F: $\left(0,\frac{3}{2}\right)$ | 3 |

Hence $Z_{max} = 19$ at $\left(\frac{7}{2},\frac{3}{4}\right)$ and $Z_{max} = 0$ at $(0, 0)$

## Case Study Answers:

**1. Answer :**

(i) (c) x+y>0

(ii) (b) (8, 12)

**Solution:**

Since (8, 12) satisfy all the inequalities, therefore (8, 12) is the point in its feasible region.

(iii) (c) (8, 12)

**Solution:**

At (0, 0), z = 0

At (16, 0), z = 352

At (8, 12), z = 392

At (0, 20), z = 360

It can be observed that max z occur at (8, 12). Thus, z will attain its optimal value at (8, 12).

(iv) (c) (8, 12)

**Solution:**

We have, x + y = 20 (i)

And 3x + 2y = 48 (ii)

On solving (i) and (ii), we get

x = 8, y = 12.

Thus, the coordinates of Pare (8, 12) and hence (8, 12) is one of its corner points.

(v) (b) The optimal solution occurs at every point on the tine joining these two points.

**Solution:**

The optimal solution occurs at every point on the line joining these two points.

**2. Answer :** Construct the following table of values of objective function:

| Corner Points | Value of Z = 4x - 6y |
|---|---|
| (0, 3) | 4 × 0 - 6 × 3 = -18 |
| (5, 0) | 4 × 5 - 6 × 0 = 20 |
| (6, 8) | 4 × 6 - 6 × 8 = -24 |
| (0, 8) | 4 × 0 - 6 × 8 = -48 |

(i) (d) (0, 8)

**Solution:**

Minimum value of Z is -48 which occurs at (0, 8).

(ii) (a) (5, 0)

**Solution:**

Maximum value of Z is 20, which occurs at (5, 0).

(iii) (b) 68

**Solution:**

Maximum of Z - Minimum of Z = 20 - (-48) = 20 + 48 = 68

(iv) (c) (0, 0), (3, 0), (3, 2), (2, 3), (0, 3)

**Solution:**

The comer points of the feasible region are O(0, 0), A(3, 0), B(3, 2), C(2, 3), D(0, 3).

(v) (d) Only first quadrant.

## Assertion and Reason Answers:

1. c) A is true but R is false.
2. c) A is true but R is false.

# Probability 13

1. **Probability** is a quantitative measure of certainty.
2. In the **experimental approach** to probability, we find the probability of the occurrence of an event by actually performing the experiment a number of times and adequate recording of the happening of event.
3. In the **theoretical approach** to probability, we try to predict what will happen without actually performing the experiment.
4. The experimental probability of an event approaches to its theoretical probability if the number of trials of an experiment is very large.
5. An **outcome** is a result of a single trial of an experiment.
6. The word **'experiment'** means an operation which can produce some well-defined outcome(s).

   There are two types of experiments:

   **i. Deterministic experiments:** Experiments which when repeated under identical conditions produce the same results or outcomes are called deterministic experiments.

   **ii. Random or Probabilistic experiment:** If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes, then it is known as a random or probabilistic experiment.

   In this chapter, the term experiment will stand for random experiment.
7. The collection of all possible outcomes is called the **sample space**.
8. An outcome of a random experiment is called an **elementary event**.
9. An event associated to a random experiment is a **compound event** if it is obtained by combining two or more elementary events associated to the random experiment.
10. An event A associated to a random experiment is said to occur if any one of the elementary events associated to the event A is an outcome.
11. An elementary event is said to be **favourable** to a compound event A, if it satisfies the definition of the compound event A. In other words, an elementary event E is favourable to a compound event A, if we say that the event A occurs when E is an outcome of a trial.
12. In an experiment, if two or more events have equal chances to occur or have equal probabilities, then they are called **equally likely events.**
13. The **theoretical probability (also called classical probability) of an event E**, written as P (E), is defined as.

$$\frac{\text{Number of outcomesfavourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

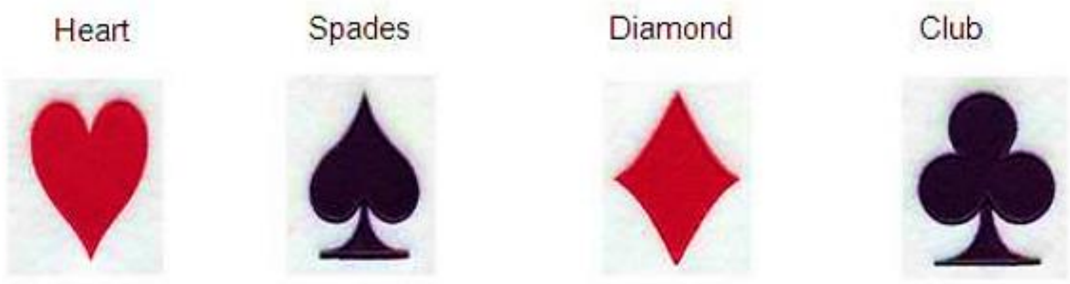
14. For two events A and B of an experiment:

    If P(A) > P(B) then event A is more likely to occur than event B.

    If P(A) = P(B) then events A and B are equally likely to occur.
15. An event is said to be sure event if it always occur whenever the experiment is performed. The probability of sure event is always one. In case of sure event elements are same as the sample space.
16. An event is said to be impossible event if it never occur whenever the experiment is performed. The probability of an impossible event is always zero. Also, the number of favourable outcome is zero for an impossible event.

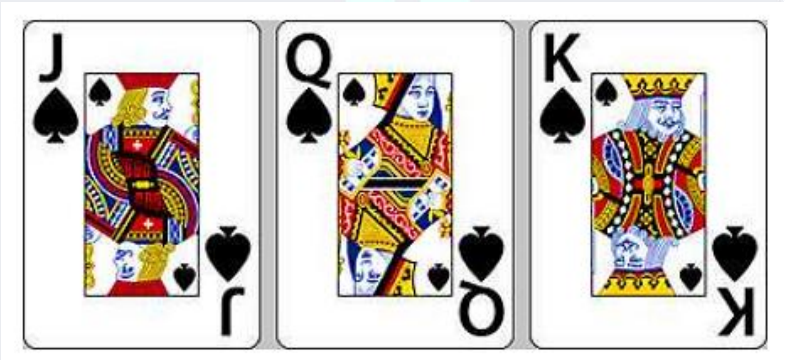
17. Probability of an event lies between 0 and 1, both inclusive, i.e., $0 \leq P(A) \leq 1$
18. If E is an event in a random experiment, then the event 'not E' (denoted by $\underline{E}$ ) is called the complementary event corresponding to E.
19. The sum of the probabilities of all elementary events of an experiment is 1.
20. For an event E, $P(\underline{E}) = 1 - P(E)$ , where the event $\underline{E}$ representing 'not E" is the complement of event E.
21. **Suits of Playing Card**

    A pack of playing cards consist of 52 cards which are divided into 4 suits of 13 cards each. Each suit consists of one ace, one king, one queen, one jack and 9 other cards numbered from 2 to 10. Four suits are named as spades, hearts, diamonds and clubs.




22. **Face Cards**

    King, queen and jack are face cards.




23. The formula for finding the geometric probability of an event is given by:

$$P(E) = \frac{\text{Mearure of the specified part of the region}}{\text{Measure of the whole region}}$$

Here, 'measure' may denote length, area or volume of the region or space.

**Class : 12th Maths**
**Chapter- 13 : Probability**

## Probability

### Random Variable

Real valued function whose domain is the sample space of a random experiment

The probability distribution of a random variable $x$ is the system of numbers $x: \; x_1 \; x_2 \; . . . . , x_n$
$P(x): \; p_1 \; p_2 \; . . . . \; p_n$ where , $pi > 0$,
$\sum_{i=1}^{n} p_i = 1, \quad i = 1.2, ...., \; n.$

### Conditional probability

The probability of the event $E$ is called the conditional probability of E given that F has already occurred, and is denoted by $P(E/F)$. Also, $P(E/F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0.$

### Properties

(i) $0 \leq P(E/F) \leq 1, P(E'/F) = 1 - P(E/F)$
(ii) $P((E \cup F)/G) = P(E/G) + P(F/G) - P((E \cap F)/G)$
(iii) $P(E \cap F) = P(E)P(F/E), P(E) \neq 0$
(iv) $P(E \cap F) = P(F)P(E/F), P(F) \neq 0$
For *eg* : if $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$ , then
$P(A/B) = \frac{p(A \cap B)}{P(B)} = \frac{\frac{4}{13}}{\frac{9}{13}} = \frac{4}{9}.$

If E and F are independent, then P E $\cap$ F = P (E)P (F), P (E|F) = P (E), P (F) $\neq$ 0 and (F|F) = P(F), P(E) $\neq$ 0.

### Variance and standard deviation

Let $x$ be a R.V. whose possible values $x_1, x_2, . . . ., x_n$ occurs with probabilities $p(x_1), p(x_2), . . . . , p(x_n)$ respectively. Let, $\mu = E(x)$ be the mean of $x$. The variance of $x$, var $(x)$ or $\sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$ or $E(x - \mu)^2$
The non-negative number
$6x = \sqrt{\text{var}(x)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)}$ is called the standard deviation of the R.V. 'X'. Also,
$\text{var}(x) = E(x^2) - [E(x)]^2$ For *eg*: $E(x) = 3$ and $E(x^2) = 10$, then var $x = 10 - 9 = 1$ and SD $= \sqrt{1} = 1$.

### Bernoulli's Trials

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :
(i)There should be a finite no. of trials.
(ii)The trials should be independent.
(iii)Each trial has exactly two outcomes: success or failure.
(iv)The probability of success remains the same in each trial.
For Binomial distribution, B $(n, p)$, $P(X = x) = ({}^{n}C_x q^{n-x} p^x, x = 0, 1, ...., n)$
$(q = 1 - p)$

### Theorem of total Probability

Let, $\{E_1, E_2 ....E_n\}$ be a partition of a sample space 'S' and suppose that each of $E_1, E_2, ....E_n$ has non-zero probability. Let 'A' be any event associated with S, then $P(A) = P(E_1)P(A|E_1) + P(E_2) P(A|E_2) + .... + P(E_n) P(A|E_n)$.

### Bayes' Theorem

If $E_1, E_2, . . . ., E_n$ are events which constitute a partition of sample space S, *i.e.*, $E_1, E_2, . . . E_n$ are pair wise dis joint and $E_1 \cup E_2 \cup .... \cup E_n = S$ and A be any event with non-zero probability,
then $P(E_i | A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$

# Important Questions

## Multiple Choice questions-

1. If P(A) = $\frac{1}{2}$, P(B) = 0, then P (A/B) is

(a) 0

(b) $\frac{1}{2}$

(c) not defined

(d) 1.

2. If A and B are events such that P (A/B) = P(B/A), then

(a) A ⊂ B but A ≠ B

(b) A = B

(c) A ∩ B = ø

(d) P (A) = P (B).

3. The probability of obtaining an even prime number on each die when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

4. Two events A and B are said to be independent if:

(a) A and B are mutually exclusive

(b) P (A'B') = [1 – P(A)] [1 – P(B)]

(c) P (A) = P (B)

(d) P (A) + P (B) = 1.

5. Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is:

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

6. If A and B are two events such that A ⊂ B and P (B) ≠ 0, then which of the following is correct

(a) P (A / B) = $\frac{p(B)}{p(A)}$

(b) P (A/B) < P (A)

(c) P (A/B) ≥ P (A)

(d) None of these.

7. If A and B are two events such that P (A) ≠ 0 and P (B/A) = 1, then

(a) A ⊂ B

(b) B ⊂ A

(c) B = ø

(d) A = ø

8. If P (A/B) > P (A), then which of the following is correct?

(a) P (B/A) < P (B)

(b) P (A ∩ B) < P (A).P(B)

(c) P (B/A) > P (B)

(d) P (B/A) = P (B).

9. If A and B are any two events such that

P (A) + P (B) – P (A and B) = P (A), then:

(a) P (B/A) = 1

(b) P (A/B) = 1

(c) P (B/A) = 0

(d) P (A/B) = 0

10. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. What is the value of E (X)?

(a) $\frac{37}{221}$

(b) $\frac{5}{13}$

(c) $\frac{1}{13}$

(d) $\frac{2}{13}$

## Very Short Questions:

1. If A and B are two independent event, prove that A' and B are also independent.

2. One card is drawn from a pack of 52 cards so that each card is equally likely to be se-lected. Prove that the following cases are in-dependent:

- A: "The card drawn is a spade"
  B: "The card drawn is an ace."
- A: "The card drawn is black"
  B: "The card drawn is a king."

3. A pair of coins is tossed once. Find the probability of showing at least one head.

4. P(A) = 0.6, P(B) = 0. 5 and P(A/B) = 0.3, then find P(A∪ B)

5. One bag contains 3 red and 5 black balls. Another bag contains 6 red and 4 black balls. A ball is transferred from first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is red.

6. Evaluate P(A ∪ B), if 2P(A) = P(B) = $\frac{5}{13}$ and P(A/B) = $\frac{2}{5}$

## Short Questions:

1. Given that A and B are two independent events such that P(A) = 0.3 and P(B) = 0.5. Find P(A/B).

2. A bag contains 3 white and 2 red balls, another bag contains 4 white and 3 red balls. One ball is drawn at random from each bag.

   Find the probability that the balls drawn are one white and one red.

3. The probabilities of A, B and C solving a problem independently are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. If all the three try to solve the problem independently, find the probability that the problem is solved.

4. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event "number is even" and B be the event "number is marked red". Find whether the events A and B are independent or not.

5. A die is thrown 6 times. If "getting an odd number" is a success, what is the probability of (i) 5 successes (ii) at most 5 successes?

6. The random variable 'X' has a probability distribution P(X) of the following form, where 'k' is some number:

$$P(X=x)=\begin{cases} k, \text{if } x=0 \\ 2k, \text{if } x=1 \\ 3k, \text{if } x=2 \\ 0, \text{otherwise} \end{cases}$$

   Determine the value of 'P.

7. Out of 8 outstanding students of a school, in which there are 3 boys and 5 girls, a team of 4 students is to be selected for a quiz competition. Find the probability that 2 boy and 2 girls are selected.

8. 12 cards numbered 1 to 12 (one number on one card), are placed in a box and mixed up thoroughly. Then a card is drawn at random from the box. If it is known that the number on the drawn card is greater than 5, find the probability that the card bears an odd number.

## Long Questions:

1. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

2. Two numbers are selected at random (with-out replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

3. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time.

4. A speaks truth in 80% cases and B speaks truth in 90% cases. In what percentage of cases are they likely to agree with each other in stating the same fact?

## Case Study Questions:

**1.** In an office three employees Vinay, Sonia and Iqbal process incoming copies of a certain form. Vinay process 50% of the forms, Sonia processes 20% and Iqbal the remaining 30% of the forms. Vinay has an error rate of 0.06, Sonia has an error rate of 0.04 and Iqbal has an error rate of 0.03.

Based on the above information, answer the following questions.

(i) The conditional probability that an error is committed in processing given that Sonia processed the form is:

   a. 0.0210
   b. 0.04
   c. 0.47
   d. 0.06

(ii) The probability that Sonia processed the form and committed an error is:

   a. 0.005
   b. 0.006
   c. 0.008
   d. 0.68

(iii) The total probability of committing an error in processing the form is:

a. 0
b. 0.047
c. 0.234
d. 1

(iv) The manager of the company wants to do a quality check. During inspection he selects a form at random from the days output of processed forms. If the form selected at random has an error, the probability that the form is NOT processed by Vinay is:

a. 1
b. $\frac{30}{47}$
c. $\frac{20}{47}$
d. $\frac{14}{47}$

(v) Let A be the event of committing an error in processing the form and let $E_1$, $E_2$ and $E_3$ be the events that Vinay, Sonia and Iqbal processed the form. The value of $\sum_{i=1}^{3} P(E_i \mid A)$ is:

a. 0
b. 0.03
c. 0.06
d. 1

**2.** Between students of class XII of two schools A and B basketball match is organised. For which, a team from each school is chosen, say $T_1$ be the team of school A and $T_2$ be the team of school B. These teams have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probability of $T_1$ winning, rawmg an osrng a game against $T_2$ are $\frac{1}{2}$, $\frac{3}{10}$ and $\frac{1}{5}$ respectively.

Each team gets 2 points for a win, 1 point for a draw and 0 point for a loss in a game.

Let X and Y denote the total points scored by team A and B respectively, after two games.

Based on the above information, answer the following questions.

i. P($T_2$ winning a match against $T_1$) is equal to:

a. $\frac{1}{5}$
b. $\frac{1}{6}$
c. $\frac{1}{3}$
d. None of these

ii. P($T_2$ winning a match against $T_1$) is equal to:

a. $\frac{1}{2}$
b. $\frac{1}{3}$
c. $\frac{1}{6}$
d. $\frac{3}{10}$

iii. P(X > Y) is equal to:

a. $\frac{1}{4}$
b. $\frac{5}{12}$
c. $\frac{1}{20}$
d. $\frac{11}{20}$

iv. P(X = Y) is equal to:

a. $\frac{11}{100}$
b. $\frac{1}{3}$
c. $\frac{29}{100}$
d. $\frac{1}{2}$

v. P(X + Y = 8) is equal to:

a. 0
b. $\frac{5}{12}$
c. $\frac{13}{36}$
d. $\frac{7}{12}$

# Answer Key

## Multiple Choice Questions-

1. **Answer:** (c) not defined
2. **Answer:** (d) P (A) = P (B).
3. **Answer:** (d) $\frac{1}{36}$
4. **Answer:** (b) P (A'B') = [1 – P(A)] [1 – P(B)]
5. **Answer:** (a) $\frac{4}{5}$
6. **Answer:** (c) P (A/B) ≥ P (A)
7. **Answer:** (a) A ⊂ B
8. **Answer:** (c) P (B/A) > P (B)
9. **Answer:** (b) P (A/B) = 1
10. **Answer:** (d) $\frac{2}{13}$

## Very Short Answer:

**1. Solution:**

Since A and B are independent events, [Given]

.-. P (A ∩ B) = P (A) . P(B) ...(1)

Now P(A'∩B) = P (B) – P (A ∩ B)

= P(B) – P (A) P( ∩ B) [Using (1)]

= (1 – P(A)) P(B) = P(A') P(B).

Hence, A' and B are independent events.

**2. Solution:**

(a) P(A) = $\frac{13}{52} = \frac{1}{4}$, P(B) = $\frac{4}{52} = \frac{1}{13}$

P(A∩B) = $\frac{1}{52} = \frac{1}{4} \cdot \frac{1}{13}$ = p(A).p(B)

Hence, the events A and B are independent

(b) P(A) = $\frac{26}{52} = \frac{1}{2}$, P(B) = $\frac{4}{52} = \frac{1}{13}$

P(A∩B) = $\frac{2}{52} = \frac{1}{26} = \frac{1}{2} \cdot \frac{1}{13}$ = P(A) . P(B)

Hence, the events A and B are independent

**3. Solution:**

S, Sample space = {HH, HT, TH, TT}

where H ≡ Head and T ≡ Tail.

∴ P (at least one head) = $\frac{3}{4}$.

**4. Solution:**

We have: P(A/B) = 0.3

$\frac{P(A \cap B)}{P(B)} = 0.3$

$\frac{P(A \cap B)}{0.5} = 0.3$

P (A ∩ B) = 0.5 x 0.3 = 0.15.

Now, P(A∪B) = P(A) + P(B) – P(A ∩ B)

= 0.6 + 0.5 – 0. 15

Hence, P (A ∪ B) = 1.1 – 0.15 = 0.95.

**5. Solution:**

P (Red transferred and red drawn or black transferred red drawn)

$$= \frac{3}{8} \times \frac{7}{11} + \frac{5}{8} \times \frac{6}{11}$$

$$= \frac{21}{88} + \frac{30}{88} = \frac{51}{88}$$

**6. Solution:**

$$P(A/B) = \frac{P(A \cap B)}{P(B)}\ P(A \cap B) = \frac{2}{11}$$

P(A∪B) = P(A) + P(B) – (A ∩ B)

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$$

## Short Answer:

**1. Solution:**

We have:

P(A)= 0.3 and P(B) = 0.5.

Now P (A ∩ B) = P(A). P(B)

[∵A and B are independent events]

= (0.3) (0.5) = 0.15.

Hence, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.5} = 0.3$

**2. Solution:**

Reqd. probability

= P (White, Red) + P (Red, White)

$$\frac{3}{5} \times \frac{3}{7} + \frac{2}{5} \times \frac{4}{7} = \frac{9}{35} + \frac{8}{35} = \frac{17}{35}$$

**3. Solution:**

Given: $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{4}$

$$\therefore P(\bar{A}) = 1 - \frac{1}{2} = \frac{1}{2}, P(\bar{B}) = 1 - \frac{1}{3} = \frac{2}{3} \text{ and } P(\bar{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Probability that the problem is solved

= Probability that the problem is solved by at least one person

$$= 1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C})$$

$$= 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

**4. Solution:**

Here, A: number is even i.e.,

A = {2,4,6}

and B: number is red i.e.,

B = {1,2,3}

$\therefore P(A)=\frac{3}{6}=\frac{1}{2}$ and $P(B)=\frac{3}{6}=\frac{1}{2}$

and,

P (A ∩ B) = P(Number is even and red) = $\frac{1}{6}$.

Thus, P (A ∩ B) ≠ P(A). P(B)

$\left[\because \frac{1}{6}\neq\frac{1}{2}\times\frac{1}{2}\right]$

Hence, the events A and B are not independent.

**5. Solution:**

Probability of getting an odd number is one 3 1

$\text{trail}=\frac{3}{6}=\frac{1}{2}=\text{p(say)}$

Probability of getting an even number in one 3

$\text{trail}=\frac{3}{6}=\frac{1}{2}=\text{g(say)}\,0|$

Also, n = 6.

(i) P (5 successes) = P(5) = $^6C_5\, q^1\, p^5$

(ii) P (at most 5 successes)

= P (0) + P(1) + ... + P(5) = 1 – P(6)

= 1 - $^6C_6\, q^0\, p^6$

$=1-\frac{1}{64}=\frac{63}{64}$

**6. Solution:**

We have: P (X = 0) + P (X = 1) + P (X = 2) = 1

⇒ k + 2k + 3k = 1

⇒ 6k = 1

Hence, $k=\frac{1}{6}$

**7. Solution:**

Read, probability $=\frac{^3C_2\times {}^5C_2}{^8C_4}$

$=\frac{3\times 10}{70}=\frac{3}{7}$

**8. Solution:**

Let the events be as :

A: Card bears an odd number.

B: Number on the card is greater than 5.

A∩B = {7, 9, 11}.

Hence, $P(A/B)=\frac{P(A\cap B)}{P(B)}$

$=\frac{3/12}{7/12}=\frac{3}{7}$

## Long Answer:

**1. Solution:**

Let the events be as:

E: Sum of numbers is 8

F: Number of red dice less than 4.

E: {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

F = {(1, 1), (2, 1), ... (6, 1), (1, 2), (2, 2), ... (6, 2), (1, 3), (2, 3), ... (6, 2) (6, 3)}

and E ∩ F = {(5, 3), (6, 2)}

$P(E)=\frac{5}{36}, P(F)=\frac{18}{36}$

and $P(E\cap F)=\frac{2}{36}$

Hence, $P(E/F)=\frac{P(E\cap F)}{P(F)}=\frac{2/36}{18/36}=\frac{2}{18}=\frac{1}{9}$

**2. Solution:**

The first five positive integers are 1, 2, 3, 4 and 5.

We select two positive numbers in 5 x 4 = 20 way.

Out of three, two numbers are selected at random.

Let 'X' denote the larger of the two numbers.

X can be 2, 3, 4 or 5.

∴ P (X = 2) = P (Larger number is 2)

{(1, 2), (2,1)} = $\frac{2}{20}$

Similarly, P (X = 3) = $\frac{4}{20}$,

P (X = 4) = $\frac{6}{20}$

and P (X = 5) = $\frac{8}{20}$

Hence, the probability distribution is:

| X | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| P(X) | 1/10 | 2/10 | 3/10 | 4/10 |
| X. P(X) | 2/10 | 6/10 | 12/10 | 20/10 |
| $X^2$ P(X) | 4/10 | 18/10 | 48/10 | 100/10 |

$\therefore \text{Mean}=\sum X\,P(X)$

$=2\times\frac{2}{20}+3\times\frac{4}{20}+4\times\frac{6}{20}+5\times\frac{8}{20}$

$=\frac{4+12+24+40}{20}=\frac{80}{20}=4$

and variance $\sum X^2\,P(x)-\left[\sum P(x)\right]^2$

$=\frac{170}{10}-(1)^2=17-1=16.$

**3. Solution:**

We have: P(A) = Probability of student A coming to school in time = $\frac{3}{7}$

P(B) = Probability of student B coming to school in time = $\frac{5}{7}$

$$\therefore P(\bar{A}) = 1 - \frac{3}{7} = \frac{4}{7} \text{ and } P(\bar{B}) = 1 - \frac{5}{7} = \frac{2}{7}$$

∴ Probability that only one of the students coming to school in time

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A)\,P(\bar{B}) + P(\bar{B})\,PB$$

[∵ A and B are independent => A and $\bar{B}$ and $\bar{A}$ and B are also independent]

$$= \left(\frac{3}{7}\right)\left(\frac{2}{7}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{7}\right) = \frac{26}{49}$$

**4. Solution:**

$$P(A) = \frac{80}{100} = \frac{4}{5} \text{ and } P(B) = \frac{90}{100} = \frac{9}{10}$$

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{4}{5} = \frac{1}{5}$$

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{9}{10} = \frac{1}{10}$$

∴ P(Agree) = P(Both speak the truth or both tell a lie)

$$= P(AB \text{ or } \bar{A}\,\bar{B})$$

$$= P(A)\,P(B) \text{ or } P(\bar{A})\,P(\bar{B})$$

$$= \left(\frac{4}{5}\right)\left(\frac{9}{10}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{10}\right)$$

$$= \frac{36}{50} + \frac{1}{50} = \frac{37}{50} = \frac{74}{100}$$

Hence, the reqd. percentage = 74%.

## Case Study Answers:

**1. Answer :** Let A be the event of committing an error and $E_1$, $E_2$ and $E_3$ be the events that Vinay, Sonia and Iqbal processed the form.

(i) (b) 0.04

**Solution:**

Required probability = P(A|$E_2$)\

$$= \frac{P(A \cap E_2)}{P(E_2)}$$

$$= \frac{\left(0.04 \times \frac{20}{100}\right)}{\left(\frac{20}{100}\right)} = 0.04$$

(ii) (c) 0.008

**Solution:**

Required probability = P(A ∩ E2)

$$0.04 \times \frac{20}{100} = 0.008$$

(iii) (b) 0.047

**Solution:**

Total probability is given by

P(A) = P($E_1$) × P(A|$E_1$) + P($E_2$) × P(A|$E_2$) + P($E_3$) × P(A|$E_3$).

$$= \frac{50}{100} \times 0.06 + \frac{20}{100} \times 0.04 + \frac{30}{100} \times 0.03 = 0.047$$

(iv) (d) 17471747

**Solution:**

Using Bayes' theorem, we have

$$P(E_1 | A) = \frac{P(E_1) \times P(A|E_1)}{P(E_1) \times P(A|E_1) + P(E_2) \times P(A|E_2) + P(E_3) \times P(A|E_3)}$$

$$= \frac{0.5 \times 0.06}{0.5 \times 0.06 + 0.2 \times 0.04 + 0.3 \times 0.03} = \frac{30}{47}$$

$$\therefore \text{Required probability} = P(E|A)$$

$$= 1 - P(E_1|A) = 1 - \frac{30}{47} = \frac{17}{47}$$

(v) (d) 1

**Solution:**

$$\sum_{i=1}^{3} P(E_i|A) = P(E_1|A) + P(E_2|A) + P(E_3|A) = 1$$

[∴ Sum of posterior probabilities is 1]

**2. Answer :**

(i) (a) $\frac{1}{2}$

**Solution:**

Clearly, P($T_2$ winning a match against $T_1$)

= P($T_1$ losing) = $\frac{1}{5}$

(ii) (d) $\frac{3}{10}$

**Solution:**

Clearly, P($T_2$ drawing a match against $T_1$)

= P($T_1$ drawing) = $\frac{3}{10}$

(iii) (d) $\frac{11}{20}$

**Solution:**

According to given information, we have the following possibilities for the values of X and Y.

| X | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|
| Y | 0 | 1 | 2 | 3 | 4 |

Now, P(X > Y) = P(X = 4, Y = 0) + P(X = 3, Y = 1)

= P($T_1$ win) P($T_1$ win) + P($T_1$ win) P(match draw) + P(match draw) P($T_1$ win)

$$=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{3}{10}+\frac{3}{10}\times\frac{1}{2}=\frac{5+3+3}{20}=\frac{11}{20}$$

(iv) (c) $\frac{29}{100}$

**Solution:**

P(X = Y) = P(X = 2, Y = 2)

= P($T_1$ win) P($T_2$ win) + P($T_2$ win) P($T_1$ win) + P(match draw) P(match draw)

$$=\frac{1}{2}\times\frac{1}{5}+\frac{1}{5}\times\frac{1}{5}+\frac{3}{10}\times\frac{3}{10}$$

$$=\frac{1}{10}+\frac{1}{10}+\frac{9}{100}=\frac{29}{100}$$

(v) (a) 0

**Solution:**

From the given information, it is clear that maximum sum of X and Y can be 4, therefore P(X + Y = 8) = 0.

❖❖